Analysis of a coupled system of partial differential equations modeling the interaction between melt flow, global heat transfer and applied magnetic fields in crystal growth

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Abstract

The present PhD thesis is devoted to the analysis of a coupled system of nonlinear partial differential equations (PDE), that arises in the modeling of crystal growth from the melt in magnetic fields.

The phenomena described by the model are mainly the heat-transfer processes (by conduction, convection and radiation) taking place in a high-temperatures furnace heated electromagnetically, and the motion of a semiconducting melted material subject to buoyancy and applied electromagnetic forces. The model consists of the Navier-Stokes equations for a newtonian incompressible liquid, coupled to the heat equation and the low-frequency approximation of Maxwell's equations. Coupling occurs at multiple levels. The liquid is electrically conducting and subject to the Lorentz force. The buoyant motion in the liquid is taken into account by a Boussinesq type modeling of thermal expansion. On the other hand, heat is produced in the electrical conductors in the furnace by the Joule effect, and by viscous friction in the liquid, where the convective heat transport has also to be taken into account. The different parts of the furnace interact with each other due to for one part to the nonlocal radiation that takes place in the transparent cavity, and cannot be neglected at the high temperatures considered; and due for the other part to the long range of action of the electromagnetic fields.

We propose a mathematical setting for this PDE system, we derive its weak formulation, and we formulate an (initial) boundary value problem that in the mean reflects the complexity of the real-life application. The well-posedness of this (initial) boundary value problem is the mainmatter of the investigation.

We prove the existence of weak solutions allowing for general geometrical situations (discontinuous coefficients, nonsmooth material interfaces) and data, the most important requirement being only that the injected electrical power remains finite. For the time-dependent problem, a defect measure appears in the solution, which apart from the fluid remains concentrated in the boundary of the electrical conductors. In the absence of a global estimate on the radiation emitted in the cavity, a part of the defect measure is due to the nonlocal radiation effects.

The uniqueness of the weak solution is obtained only under reinforced assumptions: smallness of the input power in the stationary case, and regularity of the solution in the time-dependent case.

Regularity properties, such as the boundedness of temperature are also derived, but only in simplified settings: smooth interfaces and temperature-independent coefficients in the case of a stationary analysis, and, additionally for the transient problem, decoupled time-harmonic Maxwell.

In order to prove these results and the energy estimates on which they rely, techniques are needed for handling mathematically the Lorentz force, the nonlocal radiation operators, the Joule term and in general the L^1 right-hand side in the heat equation. Some of these techniques were already available, but some of them, presented here as auxiliary results, are used for the first time in the present thesis (resp. in the publications extracted from the thesis), and fully belong to its main part.

Zusammenfassung

Hauptthema der Dissertation ist die Analysis eines nichtlinearen, gekoppelten Systems partieller Differentialgleichungen (PDG), das in der Modellierung der Kristallzüchtung aus der Schmelze mit Magnetfeldern vorkommt. Die zu beschreibenden Phenomäne sind einerseits der im elektromagnetisch geheizten Schmelzofen erfolgende Wärmetransport (Wärmeleitung, -konvektion und -strahlung), und andererseits die Bewegung der Halbleiterschmelze unter dem Einfluss der thermischen Konvektion und der angewendeten elektromagnetischen Kräfte.

Das Modell besteht aus den Navier-Stokeschen Gleichungen für eine inkompressible Newtonsche Flüssigkeit, aus der Wärmeleitungsgleichung und aus der elektrotechnischen Näherung des Maxwellschen Systems. Die Kopplung dieser Gleichungen ist vielfach. Die die elektrisch leitende Flüssigkeit beeinflussende Lorentzkraft erzeugt eine Kopplung mit den elektromagnetischen Feldern, die aus der thermischen Ausdehnung hervorgehenden Schwimmkräfte eine Kopplung zur Temperatur. Andererseits, setzen sich die Produktionsterme in der Wärmeleitungsgleichung aus dem in den elektrischen Leitern stattfindenden Joulschen Effekt und aus den Reibungsvorgängen in der Flüssigkeit zusammen. Auch der konvektive Transport in der Flüssigkeit muss in Betracht gezogen werden. Schließlich erfolgt auch eine geometrische Kopplung zwischen den verschiedenen Ofenteilen, zum Teil durch die nichtlokale Hohlraumstrahlung, zum Teil durch die weitreichende Wirkung elektromagnetischer Felder.

Wir erörtern die schwache Formulierung dieses PDG Systems, und wir stellen ein Anfang-Randwertproblem auf, das die Komplexität der Anwendung widerspiegelt. Die Hauptfrage unserer Untersuchung ist die Wohlgestelltheit dieses Problems, sowohl im stationären als auch im zeitabhängigen Fall. Wir zeigen die Existenz schwacher Lösungen in geometrischen Situationen, in welchen unstetige Materialeigenschaften und nichtglatte Trennfläche auftreten dürfen, und für allgemeine Daten. In der Lösung zum zeitabhängigen Problem tritt ein Defektmaß auf, das außer der Flüssigkeit im Rand der elektrisch leitenden Materialien konzentriert bleibt. Da eine globale Abschätzung der im Strahlungshohlraum ausgestrahlten Wärme auch fehlt, rührt ein Teil dieses Defektmaßes von der nichtlokalen Strahlung her.

Die Eindeutigkeit der schwachen Lösung erhalten wir nur unter verstärkten Annahmen: die Kleinheit der gegebenen elektrischen Leistung im stationären Fall, und die Regularität der Lösung im zeitabhängigen Fall. Regularitätseigenschaften wie die Beschränktheit der Temperatur werden, wenn auch nur in vereinfachten Situationen, hergeleitet: glatte Materialtrennfläche und Temperaturunabhängige Koeffiziente im Fall einer stationären Analysis, und entkoppeltes, zeitharmonisches Maxwell für das transiente Problem.

Der Beweis dieser Ergebnisse erfordert spezielle mathematische Techniken, beispielweise für die Behandlung der Lorentzkraft, der nichtlokalen Strahlungsoperatoren, des Jouleschen Termes und allgemein der L^1 rechten Seite in der Wärmeleitungsgleichung. Einige dieser Techniken standen schon zur Verfügung. Andere, in dieser Arbeit als Hilfsresultate dargestellte Methoden sind hier zum ersten Mal verwendet.

Contents

Chapter 1

Introduction

This thesis is mainly devoted to the investigation of a mathematical model describing crystal growth from the melt with applied magnetic fields in the three following aspects: *melt flow*, *global heat transfer*, *magnetic field distribution*.

The isothermal flow of a newtonian, incompressible, viscous, electrically conducting fluid subject to electromagnetic forces can in most situations be described with the partial differential equations

$$\rho\left(\frac{\partial v}{\partial t}+(v\cdot\nabla)v\right)=-\nabla p+\operatorname{div}(2\,\eta\,Dv)+j\times B+f\,, \tag{1.1}$$

$$\operatorname{div} v=0\,, \tag{1.2}$$

where ρ is the mass density, η the dynamic viscosity, v the fluid velocity, p the fluid pressure and $D\,v$ the rate of strain. The expression $j\times B$ denotes the electromagnetic force, and f denotes the gravity.

In the context of low-frequency applications, the electromagnetic fields in general satisfy

$$\operatorname{curl} E+\frac{\partial B}{\partial t}=0\,, \tag{1.3}$$

$$\operatorname{curl} H=j\,, \tag{1.4}$$

$$\operatorname{div} B=0\,, \tag{1.5}$$

$$B=\mu\,H,\qquad D=\mathfrak{e}\,E\,, \tag{1.6}$$

where E is the electric field strength, B the magnetic induction, H the magnetic field strength, j the electric current density, D the electric displacement, and the parameters μ and $\mathfrak{e}$ respectively denote the magnetic permeability and the electrical permittivity of the medium. To close the problem, one further relation is needed

$$j=\mathfrak{s}\,(E+v\times B)\qquad\text{in electrical conductors,} \tag{1.7a}$$

$$\operatorname{div} D=0\qquad\text{in nonconducting materials,} \tag{1.7b}$$

where $\mathfrak{s}$ denotes the coefficient of electrical conductivity, and where the density of free charges in the right-hand side of (1.7b) was put to be zero.

In a non-isothermal situation, it is to expect that the material properties η, $\mathfrak{s}$, as well as the force f of gravity, will depend significantly on temperature. Actually, due to thermal expansion, the fluid can no longer be regarded as incompressible. For investigations concerning thermal convection in liquids, it is however often possible to reasonably assume that the fluid remains incompressible in the mean, and to approximate the fluid motion by only modeling the thermal dependence of gravity according to

$$f = f(\theta) = \rho_{\mathrm{Ref}}\,(1 - \alpha\,(\theta - \theta_{\mathrm{Ref}}))\,, \tag{1.8}$$

with a characteristic tempererature θ_{Ref}, a characteristic density ρ_{Ref} and the coefficient of thermal expansion α (Boussinesq approximation).

The temperature distribution has also to be modeled. Assuming a linear Fourier law for the heat flux, the absolute temperature θ satisfies an energy balance in the form

$$\rho\, c_V\,\left(\frac{\partial\theta}{\partial t} + v\cdot\nabla\theta\right) = \mathrm{div}(\kappa\,\nabla\theta) + \frac{|j|^2}{\mathfrak{s}} + 2\,\eta\,Dv\,:\,Dv\,, \tag{1.9}$$

where θ denotes the absolute temperature, ρ denotes the mass density, c_V is the specific heat at constant volume, and κ is the heat conductivity of the medium.

The system of magnetohydrodynamic (1.1),..., (1.7b) complemented by (1.8) and (1.9) represents a system of eight partial differential relations for the unknowns v, p, E, B, H, j, D, θ. In order to formulate the correct boundary conditions to which these functions and fields are subject in applications, we have to take into account that a crystal growth furnace has in general a very complex structure with heterogeneous materials.

For the velocity of the fluid, we will impose adherence boundary condition

$$v = v_g\,, \tag{1.10}$$

with the given velocity v_g of the (for instance rotating) vessel.

At interfaces that separate heterogeneous materials, the electromagnetic fields satisfy

$$[E \times \vec{n}] = 0, \quad [H \times \vec{n}] = 0, \quad [B\cdot\vec{n}] = 0, \tag{1.11}$$

where the square brackets denote the jump of quantity at the surface, and $\vec{n}$ is a unit normal to the surface.

At interfaces between opaque materials, we assume the continuity of the heat-flux. For high temperatures applications, in which the effect of heat radiation cannot be neglected, the jump of the heat flux at the interface between an opaque and a transparent material is given by

$$[-\kappa\,\frac{\partial\theta}{\partial\vec{n}}] = R - J\,, \tag{1.12}$$

where R denotes the radiation outgoing from the surface, and J the incoming radiation. These functions are additional unknowns, that have to be determined from the relations

$$R = \sigma\,\epsilon\,\theta^4 + (1-\epsilon)\,J\,, \tag{1.13}$$

$$J = K(R)\,, \tag{1.14}$$

where ϵ is called the emissivity of the surface and attains values in $[0, 1]$, σ denotes the Stefan-Boltzmann constant, and K is a given integral operator that we will specify below. Note that the radiation boundary conditions (1.12), (1.13), (1.14) actually lead to the following integro-differential problem at interfaces between opaque and transparent materials:

$$\begin{aligned}&[-\kappa \frac{\partial \theta}{\partial \vec{n}}] = (I - K)(R),\\&(I - (1 - \epsilon)\, K)(R) = \epsilon\, \sigma\, \theta^4\,.\end{aligned} \tag{1.15}$$

No analytical results are available concerning the full system (1.1), (1.2), (1.3), (1.4), (1.5), (1.6), (1.7a), (1.7b), (1.8), (1.9), subject to the boundary conditions (1.10), (1.11) and (1.15). We can mention four facts that make this problem difficult to be handled analytically[1]:

(1) No obvious regularity argument for the system of MHD allows to prove that the right-hand side of the heat equation is better than only integrable.

(2) There is no theory available for handling the nonlocal radiation boundary operators in relation with only integrable right-hand sides.

(3) In the generalized setting of the Maxwell system, the Lorentz force $j \times B$ as right-hand side of the Navier-Stokes equations is *a priori* not better than L^1.

(4) The temperature-dependent buoyancy forces introduced by (1.8) disturb the global energy balance.

To deal with these four challenges, we prove in the thesis *preparatory results*. These are on the one hand results concerning the nonlocal radiation operators (smoothing properties, coercivity inequalities), sufficiently fine to handle a right-hand side of the heat equation that might be only integrable. On the other hand, we present results concerning embedding properties of the spaces used in the weak setting of Maxwell's equations to obtain the higher integrability of the Lorentz force.

We are then able to present *existence* results. We prove the existence of stationary weak solutions in the case of stationary data, and the existence of weak solutions with defect measure for general time-dependent settings. In view of the state of the research, defect measures seem to be unavoidable in the analysis of transient coupled problems with quadratic energy dissipation[2].

The main advance consists in our ability to treat heat radiation, which cannot be done with the available techniques. But we would like to draw the attention on other features of our results that go beyond the simple adding of heat radiation to already solved problems. Included are existence results in the case that more than two materials with heterogeneous electromagnetic properties are in contact with each other, and for

[1] In the second chapter of the thesis, once we have introduced the functional setting of the equations, we will describe more precisely our main results, and put them in the context of related knowledge.

[2] The fact that only a supsolution is attained in the heat balance is sometimes even justified by physical considerations.

interfaces with corners. Such results are at this time not available even for the isothermal MHD. Also not available is an existence result with dissipative heating *and* no truncation of the buoyancy forces.

We prove the *uniqueness* of the weak solution for small data in the stationary case, and for strong solutions in the time-dependent setting. These uniqueness results are standard, that means, expected in view of the knowledge available on related hydrodynamic problems. However, this expectedness does not make the derivation more easy, and here also we have to use special techniques.

Finally, *qualitative properties* of the weak solutions are derived, concerning mainly the estimates of the solutions in terms of the data of the problem, the concentration behavior of the defect measure in the case that the dissipative heating is neglected in the fluid, and regularity properties in simplified or decoupled settings.

Structure of the thesis In the **chapter** 2, we motivate and introduce the model starting from a problem that has stimulated recent research in the area of crystal growth: the possibility to *influence and stabilize the flow of a melted metal subject to a temperature gradient*, i.e. to buoyancy, with the help of magnetic fields. In the **chapter** 3, we introduce a concept of weak solutions, a vector space setting for the functionalanalytic method, and we summarize the state of the research. Our main results and statements are commented in greater details also in the second chapter.

The **chapter** 4 is an essential preparation to the analysis of the (initial) boundary value problems for (1.1),..., (1.7b) with (1.8) and (1.9). It is concerned with the properties of the nonlocal boundary operators introduced by the modeling of heat radiation. The **chapter** 5 investigates the problem of the higher integrability of the Lorentz force $j \times B$ in the Navier-Stokes equations.

The **chapter** 6 deals with the boundary value problem for the stationary system. We first prove a general existence result assuming that the buoyancy forces f given by (1.8) can be truncated. We then show in a second section that the existence theory can also deal with (1.8) without truncation by assuming that the coefficient of termal expansion α is comparatively small. This is a prerequisite of the Boussinesq model. Uniqueness can be proved for small data only, that is, for small input electrical power (third section). In smooth domains and with constant coefficients, regularity is obtained (fourth section).

The **chapter** 7 is devoted to the initial boundary value problem for the time-dependent system. We obtain a general existence result with defect measure for truncated buoyancy forces f given by (1.8). This result is preserved in the case (1.8) for small coefficient of thermal expansion α. Uniqueness is obtained in the third section for strong solutions only. At the cost of further simplifying the model, regularity results can be proved (fourth section). The first part containing the main text of the thesis ends with this chapter.

In the **appendix** A, we have for convenience gathered in a toolbox some formula, some technical estimates and statements used through the thesis. In the **appendix** B, the equations are scaled with typical quantities in order to obtain dimensionless relations. The most important dimensionless numbers and their relationship to the physical phenomena described by the model are recalled. In the **appendix** C, we propose some complements to the model to more precisely describe heat and mass transfer in the three phase system melt-crystal-gas.

Chapter 2

Motivation. Crystal growth from the melt in magnetic fields

2.1 Czochralski's method in crystal growth. The melt instability

To growth single crystals means to induce recrystallization of polycrystalline materials around a single crystal seed, which can be realized with several methods. In Czochralski's method the crystals are *pulled* from the *melt* of the polycrystalline material, an idea nowadays realized at very large scales by the semiconductor industry[1].

Figure 2.1 represents a typical high-temperatures furnace for the growth of single crystals[2]. The polycrystalline semiconducting material is filled in the crucible. Once the material has been melted, the pull rod in the upper part of the furnace is used to dip a single crystal seed into the melt. The art of crystal growth consists in adjusting the growth parameters (input power, crystal rotation, pulling velocity), and in lifting the seed in such a way that it goes on to support a meniscus of liquid. Recrystallization can occur through cooling at the contact of the colder gas phase. The rotation of the pull rod ensures the circular shape of the crystal, the so-called *ingot form*[3].

For the production of reliable electronic devices, crystals of high quality are required. Determining for the crystal quality are chiefly the thermodynamical parameters at the crystallization interface, such as for instance the shape of the free phase boundary or the radial temperature gradient responsible for thermal stresses in the crystal. To under-

[1]Detailed descriptions of Czochralski's method from the point of view of applied mathematicians can be found for instance in Voigt [2001], page 1-5 or in Breuer [2002], page 354-357.

[2]Induction heating is the more commonly employed heating system, the inductor consisting of several copper coils ring located outside of the furnace. For the apparatus depicted here, the fact that the coil rings are located inside of the furnace, and that resistive heating is employed, constitutes an originality.

[3] In order to understand the scopes of industrial crystal growth, a few numbers might be helpful. The furnace of Figure 2.1 is roughly $1m$ high and $20cm$ large and is used for the growth of gallium-arsenide single crystals. Industrial crystals usually have a diameter between $200mm$ and $300mm$ (in the case of silicon crystals). The growth of high quality crystals of $400mm$ diameter is nowadays regarded as a realistic hope. The melting temperatures are high: $1685K$ for Si, $1511K$ for gallium-arsenide. The pulling process is relatively slow, of the order of $30-150mm/h$ in the case of silicon growth. The crystal rotation has to be adjusted, but in general it is of the order of a few rotations pro minute.

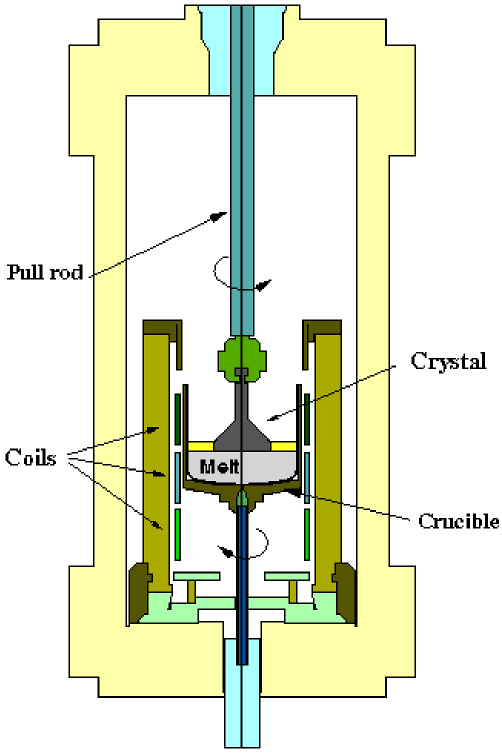


Figure 2.1: Schematic cross-sectional representation of a growth arrangement of the Institute of Crystal Growth (IKZ) Berlin.

stand the mechanisms that lead to defect formation in the growing crystal and to find appropriate mathematical models is at the present time a field of intense research [4].

The *melt motion* in the crucible is also a factor of decisive influence. The flow in the melt is principally due to thermal convection, originating from the temperature difference between the surface of the melt and the warmer walls of the crucible. The figure 2.2 below roughly depicts the convective flow pattern.

A liquid subject to a temperature gradient is *thermally unstable*[5], and this is assumed to be responsible for the formation of inhomogeneities in the crystal lattice[6]. Because of its damping effect on thermal instability, a rotation of the crucible, opposite to the crystal rotation, is part of the classical Czochralski method. However, it has to remain relatively slow, of the order of a few rotations pro minute. The large melt dimensions used in industry [7] are also a factor that diminishes the influence of the viscous forces in

[4]The thesis Voigt [2001] makes defect formation in the crystal to one of its main concern. The interested reader can find a lot of related references therein

[5]We cannot define instability better than with the words of Chandrasekhar [1981], page 1-2: "We ask: if the system is disturbed, will the disturbance gradually die down, or will the disturbance grow in amplitude in such a way that the system progressively departs from the initial state and never reverts to it? In the former case, we say that the system is *stable* with respect to the particular disturbance, and in the latter case, we say that it is *unstable*."

[6]see Voigt [2001], page 10

[7]At this time, $800mm$ is a common diameter of the crucible. Note that for the growth of one silicon crystal of $300mm$ diameter, up to $300kg$ of melted material are necessary!

the flow.

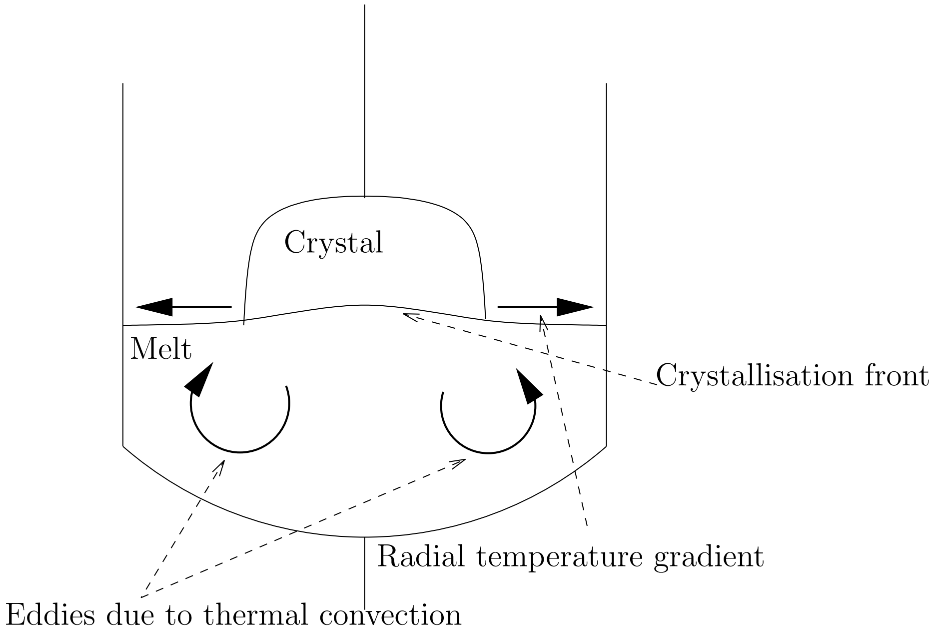


Figure 2.2: The radial temperature gradient and the convective flow pattern, after Voigt [2001].

The application of magnetic fields generates additional forces that can be used to influence the motion of electrically conducting fluids, in particular of melted metals characterized by their low viscosity. *Magnetohydrodynamics* (MHD), also called *hydromagnetics*, is a part of plasma physics, in which approximations are developed appropriated for the investigation of the interactions between fluids and electromagnetic fields in the area of technology. The MHD approximation can be identified with the basic equations (1.1), (1.2), (1.3), (1.4), (1.5), (1.6), (1.7a), (1.7b) (see Cap [1972], Jackson [1999], or many other textbooks).

Due to the electromagnetic force $j \times B$ in the equation (1.1), magnetic fields *can profoundly affect* the motion of electrically conducting fluids. This basic fact has been demonstated experimentally, and theoretically at the example of particular solutions to the system of MHD[8].

In a considerable number of examples, the effect of the magnetic field amounts to *increasing the viscosity* of the fluid. Magnetic fields have proven their ability to damp unstable flows originating from thermal convection[9]. A basic explanation of this damping effect on thermally instable fluids is that the resistivity of the fluid acts as a second viscosity. The Joule effect increases the quantity of heat produced in the fluid [10], so that thermal instability can only set in at higher temperature gradients.

[8]See for example the detailed analysis of the *Hartmann solution* in Cap [1972], volume III, page 31-37, or in Jackson [1999].

[9]The reference Cap [1972], volume III, page 128-135, describes how a steady state magnetic field is used to increase the stability region of the convective flow in a Bénard cell. A more detailed analysis of the same problem can be found in Chandrasekhar [1981], chapter I-VI, in particular chapter III.

[10]Cap [1972], or Chandrasekhar [1981], page 160

In the particular area of crystal growth, detailed investigations of different types of magnetic fields and their specific effect are at the center of intense research (see Friedrich et al. [1999], Lechner et al. [2007] and related publications). Mostly used in the area of technology are steady states magnetic fields, or time-harmonic magnetic fields of the form

$$H(t,\, x) = H_0(x)\, \sin(\omega\, t + \Phi(x))\,,$$

where H_0 is the amplitude, ω the angular frequency, and Φ the phase-shift. Diversity in the type of magnetic field is as well realized changing the disposition of the magnetic coils around the furnace.

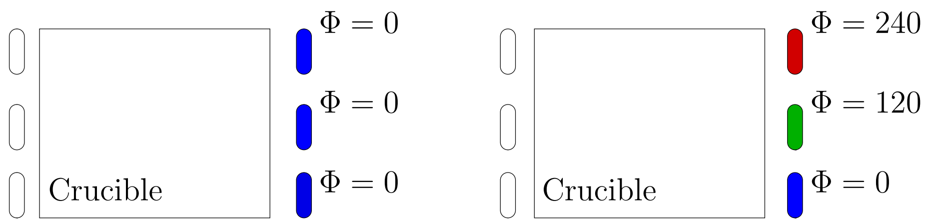


Figure 2.3: Three coil rings in axial disposition. Left: alternating magnetic field ($\Phi = 0$). Right: Traveling magnetic field. (After J. Friedrich's lectures at the IKZ Berlin, May 2006).

At the present time, it is generally accepted that magnetic fields can be used to significatively influence important parameters in crystal growth from the melt (such as the bending of the solidification interface, the temperature oscillations in the melt), or to impose specific flow patterns. But a lot of questions remain to be answered concerning the global influence of a magnetic field on Czochralski crystal growth, especially its precise influence on the crystallization process.

On the other hand, the theoretical practicability of the melt stabilization by magnetic fields is not yet equivalent to technical feasibility, let alone to a rentable use in industry. A field sufficiently strong to show a positive influence on the melt is to realize only at the cost of additional input power. Some of these open questions have been recently investigated in the project *Krist$\widetilde{MAG}$* (see http://www.kristmag.com). In this project, a technological innovation was proposed to make *traveling magnetic fields* for Czochralski crystal growth attractive for the industry (see Rudolph [2007]): the induction coils that usually surround the furnace are replaced by a resistance heater in the furnace, specifically designed to at the same time generate a traveling magnetic field. In this way, the power used to heat the furnace is cleverly redirected to give control possibilities on the melt.

All these theoretical investigations or technological innovations need the support of verification. Since experiments in high-temperatures situations are difficult and costly, a wide interesting field of challenging problems for applied mathematics, especially in the fields of mathematical modeling, of numerical mathematics (simulations) and of optimization, is opened (cp. Voigt [2001], Klein et al. [2004], Meyer [2006], Meyer et al. [2006]).

These problems are not less interesting and challenging from the point of view of applied analysis. In the present thesis, we investigate a 3D-model that aims at describing the influence of applied magnetic fields on the melt flow and on the global temperature distribution in a high-temperatures furnace of the type 2.1. In the limited scope of the thesis, some of the main difficulties of a complete modeling and analysis of Crystal growth from the melt, for instance the growth of the crystal itself, the phase transition melt-crystal will not be considered. However, the reader can consult the appendix C for some complements to the model.

2.2 Describing melt flow and global heat transfer in a crystal growth furnace

The model for the melt flow that we propose here essentially follows Voigt [2001]. Global heat transfer is modeled with an approach similar to Klein et al. [2004] for computing the heat sources from the Maxwell equations. The model for heat radiation is of wide use in crystal growth and is also described in Klein et al. [2004], Voigt [2001]. Our main references for modeling the magnetic field is the book Bossavit [2004]. In this paragraph, we do not claim to do modeling work in the proper sense, but just to present equations that many experienced authors are using to describe crystal growth processes.

2.2.1 The model for the fluid flow

The melt flow is governed by the Navier-Stokes equations for a compressible, viscous, electrically conducting and heat-conducting fluid. However, it is widely accepted that the model can significantly be simplified for the study of thermal (natural) convection in liquids. According to Boussinesq's approximation (see for instance Gray and Giorgini [1976] for a general description), it is possible to assume in such cases that the density variations in the fluid remain so small that the fluid still can be regarded as *incompressible in the mean*. The velocity v and the pressure p in the melt are consequently assumed to satisfy the Navier-Stokes equations in the form

$$\rho_1 \left(\frac{\partial v}{\partial t} + (v \cdot \nabla) v \right) = -\nabla p + \operatorname{div}(2\,\eta\, D\, v) + F\,,$$

$$\operatorname{div} v = 0\,, \tag{2.1}$$

where the reference mass density ρ_1 of the fluid is a given constant, where η denotes the dynamic viscosity of the fluid, and $D\,v$ is the rate of strain tensor, with the notation

$$D\,v = D_{i,j}\,(v) := \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \quad (i,j = 1, \ldots, 3)\,, \tag{2.2}$$

$$D(u,\,v) := D\,u \,:\, D\,v := D_{i,j}\,(u)\, D_{i,j}\,(v)\,. \tag{2.3}$$

Here and throughout this thesis, we use the convention that repeated indices imply summation over 1, 2, 3.

The external force F is twofold. On the one hand the melt flow in Czochralski crystal growth is mainly driven by buoyancy. Denoting by ρ the mass density of the fluid, and using linear expansion, we can write for the thermal expansion of the fluid

$$\rho = \rho(\theta) = \rho_1 \left(1 - \alpha \left(\theta - \theta_1\right)\right), \tag{2.4}$$

where α is the thermal expansion coefficient of the fluid, and θ_1 the reference temperature. Boussinesq's model consists in setting

$$f = f(\theta) := \rho(\theta) \, \vec{g} \,, \tag{2.5}$$

where f is the gravity, and $\vec{g}$ is the fixed vector of earth acceleration.

On the other hand, since the electrically conducting fluid is in presence of a magnetic field, it is subject to the Lorentz force $j \times B$, where j denotes the vector of the current density and B the vector of the magnetic induction. Therefore, the resulting external force is given by

$$F = f(\theta) + j \times B \,, \tag{2.6}$$

2.2.2 The model for global heat transfer

The main heat-transfer phenomena in Crystal growth from the melt are

Heat conduction;

Heat convection, in the melt and in the gas that fills the transparent cavity in the furnace;

Heat radiation in the transparent cavity.

There is a general agreement to consider that heat transport in the gas is dominated by radiation, so that the heat convection in the gas can be neglected in the model. Not taking into account the heat convection in the gas is a useful simplification, since otherwise one would need to describe also the gas motion[11]. The global temperature distribution in the furnace is thus governed by the heat equation

$$\rho \, c_V \left(\frac{\partial \theta}{\partial t} + v \cdot \nabla \theta \right) = \operatorname{div}(\kappa \, \nabla \theta) + f \,, \tag{2.7}$$

where θ denotes the absolute temperature, ρ denotes the mass density, c_V is the specific heat at constant volume, and κ is the heat conductivity of the medium. Since we neglect the gas convection, we make the simplifying assumption that $v \neq 0$ only in the melt.

The heat sources f result, on the one hand, from the Joule effect in the electrical conductors. Elsewhere, heat is only producted by viscous friction in the fluid. Therefore

$$f = \frac{|j|^2}{\mathfrak{s}} + 2 \, \eta \, D(v, \, v) \,, \tag{2.8}$$

where $\mathfrak{s}$ denotes the electrical conductivity, j the density of electrical current supported in the conductors, and v the velocity supported in the melt.

[11]The non-participation assumption on the gas can become critical as the pressure increases in the gas atmosphere, or for certain growth arrangement, see Voigt [2001], page 10 and the therein cited references.

Heat radiation Heat radiation is emitted at the surfaces of the opaque bodies that are located in the transparent cavity inside of the furnace. On this surface, the energy balance takes the form

$$\left[-\kappa \frac{\partial\theta}{\partial\vec{n}}\right] = R - J\,, \tag{2.9}$$

where R is the radiosity (outgoing radiation) and J is the incoming radiation. The relation (2.9) means that the outgoing conductive heat flux has to balance the energy brought to the surface by radiation. On the other hand, a simple constitutive relation is given by

$$R = \epsilon\,\sigma\,\theta^4 + (1-\epsilon)\,J\,, \tag{2.10}$$

which means that the outgoing radiation is the sum of the radiation emitted according to the Stefan-Boltzmann law, and of the reflected radiation. In (2.10), the function ϵ, that attains value in $[0,\,1]$, is the emissivity of the body, and σ denotes the Stefan-Boltzmann constant. Following most modeling approaches for global heat transfer Klein et al. [2004], Voigt [2001] in crystal growth, we assume that all materials involved are *diffuse grey*. Accordingly, the material parameters *emissivity* and *reflexivity* depend neither on the angle of incidence nor on the wavelength. We in addition assume that all materials are opaque except for the gas, and that no interaction takes place between gas and radiation.

One need to obtain a second constitutive relation between R and J. For two arbitrary points z, y on the surface of the transparent cavity that can see each other, the part of the radiation outgoing at the point y that attains the point z, that we denote by $j_y(z)$, is given by the inverse square law

$$j_y(z) = \frac{\vec{n}(z)\cdot(y-z)\ \vec{n}(y)\cdot(z-y)}{\pi\,|y-z|^4}\,R(y)\,,$$

where $\vec{n}$ denote a unit normal to Σ. In order to obtain an expression for the total radiation $J(z)$ incoming at point z, we have to sum up all contributions $j_y(z)$ over the boundary of the transparent cavity where z is located.

Denoting this cavity by Ω_0, and its boundary by Σ, we at first need to introduce the *visibility function* $\Theta:\ \Sigma\times\Sigma\to\{0,1\}$, which penalizes the radiation whenever the line $]z,\,y[:= conv(z,\,y)\setminus\{z,\,y\}$ crosses an opaque obstacle:

$$\Theta(z,y) = \begin{cases} 1 & \text{if }]z,y[\,\subset\Omega_0\,,\\ 0 & \text{else.}\end{cases} \tag{2.11}$$

We then introduce the so-called *view factor*, which for pairs of points $(z,\,y)\in\Sigma\times\Sigma$ is given by

$$w(z,y) := \begin{cases} \dfrac{\vec{n}(z)\cdot(y-z)\ \vec{n}(y)\cdot(z-y)}{\pi|y-z|^4}\,\Theta(z,y) & \text{if } z\neq y\,,\\ 0 & \text{if } z=y\,.\end{cases} \tag{2.12}$$

We obtain the total incoming radiation at $z\in\Sigma$ by setting

$$J = K(R)\qquad \text{on } \Sigma\,. \tag{2.13}$$

with the linear integral operator K is defined by

$$(K(R))(z) := \int_{\Sigma} w(z,y)\, R(y)\, dS_y \quad \text{ for } z \in \Sigma\,. \tag{2.14}$$

The relations (2.10) and (2.13) are equivalent to the *radiosity* equation

$$(I - (1-\epsilon)\, K)(R) = \epsilon\, \sigma\, \theta^4\,, \tag{2.15}$$

which is an integral equation of the second kind posed on the boundary of the transparent cavity. Under mild assumptions on the geometry and data (cp. Lemma 4.1.5, (4)), that we will make throughout the thesis, the solution operator of the equation (2.15) is well-defined, and the quantities R, J can be eliminated from the formulation of the problem. Introducing the linear operator

$$G := (I - K)\, (I - (1-\epsilon)\, K)^{-1}\, \epsilon, \tag{2.16}$$

it can be shown from (2.10) and (2.13) that the boundary condition (2.9) can equivalently be expressed

$$\left[-\kappa\, \frac{\partial \theta}{\partial \vec{n}} \right] = G(\sigma\, \theta^4)\,, \tag{2.17}$$

where only the unknown θ is involved.

2.2.3 The model for electromagnetics

In crystal growth *without* additional applied magnetic fields, a modeling of the electromagnetic inductive and/or resistive heating system is necessary to compute the heat sources (see Klein et al. [2004], Lechner et al. [2007] and the references therein). However, it is often satisfactory to neglect the interaction of the fields generated in this way with the fluid motion. This is of course not anymore the case if such interaction is at the core of the investigation.

A fundamental difficulty for the computation of magnetic fields is their wide range of action. It is seldom realistic to assume that the applied magnetic field is confined to the region of interest for the computation of temperature, the furnace, let alone of the velocity, the crucible. However, since it is clearly neither necessary to consider extension of the electromagnetic fields to the entire space, we assume that they extend to some *bounded region* which may be *larger than the furnace*. This assumption is central in most numerical models (see Bossavit [2004], Ch. 5 or Monk [2003], 13.5).

In this region, the electric field E and the magnetic induction B satisfy Faraday's law of induction

$$\operatorname{curl} E + \frac{\partial B}{\partial t} = 0\,. \tag{2.18}$$

Magnetohydrodynamics, or low-frequency approximation of Maxwell's equations, means that Ampère's law

$$\operatorname{curl} H = j\,, \tag{2.19}$$

is assumed to be valid for the vector H of the magnetic field strength and the current density j. The magnetic induction B satisfies the so-called Gauss law

$$\operatorname{div} B = 0\,. \tag{2.20}$$

In the electrical conductors, Ohm's law is valid in the form

$$j = \mathfrak{s}\,(E + v \times B)\,, \tag{2.21}$$

where $\mathfrak{s}$ denotes the electrical conductivity, and where we assume for simplicity that v is supported in the melt. In the nonconductors, the vector field D of electric displacement has to satisfy the conservation of charge, that means, in the absence of free charges,

$$\operatorname{div} D = 0\,. \tag{2.22}$$

We need a constitutive relation between B and H, as well as between E and D. We consider only linear media, that is

$$B = \mu\, H, \quad D = \mathfrak{e}\, E\,, \tag{2.23}$$

where μ is the magnetic permeability, and $\mathfrak{e}$ is the electrical permittivity.

We now have to model the presence of a current source in some parts of the conductors. We can denote by $\tilde{\Omega}_{c_0}$ the conductors where a current source is acting. Typically, these are magnetic coils that surround the furnace. In the case of picture 2.1, the coils are disposed inside of the furnace. We discuss two possibilities for modeling the current sources.

First model: In the first model one considers that the current is *imposed* by the current source in $\tilde{\Omega}_{c_0}$. This model is the natural one if the applied current is a *direct current*, but is also widely used to approximate technical applications with alternating current (see Bossavit [2004], Ch. 5), for example in the case that $\tilde{\Omega}_{c_0}$ is an inductor that does not belong to the furnace. We have $\operatorname{curl} H = j_g$ in $\tilde{\Omega}_{c_0}$, where j_g denotes the known density of the given current.

Second model: In the second model, one considers that the induction exerted by the system on the conductors $\tilde{\Omega}_{c_0}$ is not negligible. Therefore, it is not possible to regard the current as being imposed therein.

Observe that from (2.18) and (2.20), it follows that

$$E = -\frac{\partial A}{\partial t} + \nabla \chi\,, \tag{2.24}$$

with a magnetic potential A of B, and a scalar potential χ.

In the second model, it is assumed that the part $\mathfrak{s}\,\nabla\chi$ of the current is originating from an applied voltage, and that only this part of the current can be considered as imposed. Therefore

$$\mathfrak{s}\,\nabla\chi \sim j_g \quad \text{in }]0,T[\times\tilde{\Omega}_{c_0}\,. \tag{2.25}$$

where j_g denotes the known density of the given current. It follows that (3.25) and (3.28) have to be written in the form

$$\operatorname{curl} H = \mathfrak{s}\,\left(-\frac{\partial A}{\partial t} + v \times B\right) + j_g\,, \tag{2.26}$$

with j_g supported in $\tilde{\Omega}_{c_0}$. The potential A is related to B by the relation $\operatorname{curl} A = B$. The choice of A can be fixed by an additional condition called gauge.

2.3 Brief discussion of the model

The model proposed in the paragraph 2.2 relies on different approximations and simplifications of the general equations governing the motion of a compressible, viscous, heat-conducting plasma (see Cap [1972], volume I, or Jackson [1999]). It is beyond the purpose of this thesis to justify the use of this model in every detail. However, it is useful to point at those features of the model that have *a decisive influence on the mathematical nature of the problem*[12].

Approximation in general should be justified mathematically, that means by deriving precise *error estimates* satisfied by the approximate solutions. Rigorous justifications are however difficult, since the solution of the more complex problem must be at hand. In practice, the use of approximations more often relies on *dimensional analysis*, that is on an appropriate scaling of the equations (see the appendix B), or even on *experience*.

MHD approximation The MHD approximation of a plasma in particular consists in replacing the Maxwell equation

$$-\frac{\partial D}{\partial t} + \operatorname{curl} H = j\,,$$

by Ampère's law (2.19). This is nothing else than the well-known 'approximation of electrotechnics', or the 'low-frequency approximation'. In the context of an industrial application, we generally expect that the use of this approximation is justified. Of course, examples can be given where this 'rule' is violated (see for example Bossavit [2004], 4.3).

The low-frequency approximation has been justified mathematically as the asymptotic limit $\omega \to 0$ of Maxwell's equations, where ω denotes the frequency of the applied current (see for example Bossavit [2004], 4.3). This approximation transforms the hyperbolic system of Maxwell's equations into a *parabolic* system.

Boussinesq approximation The Boussinesq approximation of a compressible fluid consist in replacing the continuity equation

$$\operatorname{div} v = \frac{-1}{\rho}\left(v \cdot \nabla\rho + \frac{\partial \rho}{\partial t}\right)\,, \tag{2.27}$$

by the incompressibility condition in (1.2). At the same time, the thermal expansion of the fluid and the buoyancy effects are taken into account by the linear expansion (2.4), and the ansatz (2.5).

The Boussinesq approximation is a generally well-accepted, widely used model to analyse natural convection in liquids. A lot of contributions are concerned with determining its range of validity. The Boussinesq approximation has also been justified mathematically in the work of Feireisl. In Feireisl and Novotnỳ [2007] (among others), it is shown that the Boussinesq equations are the asymptotic limit of the full compressible Navier-Stokes system as the *Mach number* (see the appendix B) tends to zero.

[12]Note also that the model claims validity only for the description of the influence of the magnetic field on the melt flow and the global temperature distribution in the furnace. The reader interested in a more refined modeling of the growth process can consult the appendix C below, or the literature.

Though it simplifies matters by allowing to approximate compressible flows with simpler equations, the Boussinesq approximation leads also to specific mathematical difficulties. For example, the total work done by the gravitational force

$$\int_0^t \int_{\Omega_1} \rho(\theta)\, \vec{g} \cdot v\,,$$

would vanish if the mass conservation (2.27) was satisfied exactly, but in general does not vanish for a solenoidal velocity field v.

Approximation of the current sources by a given current density The assumption that the current is imposed in the conductor $\tilde{\Omega}_{c_0}$, or even the relation (2.26), are approximations of an unknown current density that would have to be computed from data such as applied voltage, or input power (cp. with an approach such as Klein et al. [2004]).

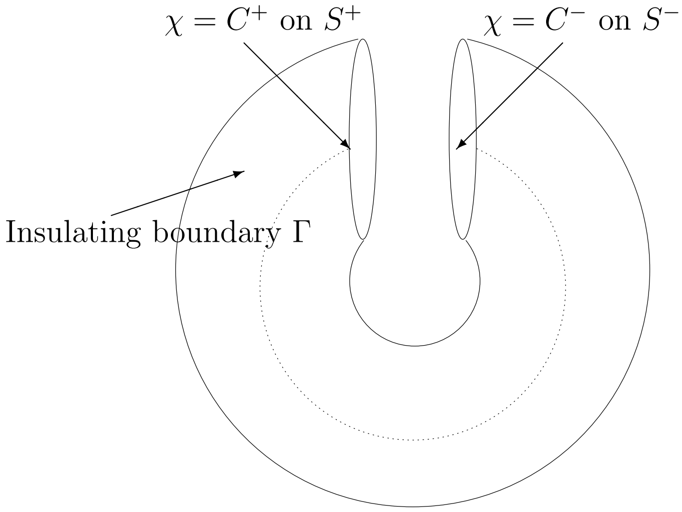


Figure 2.4: The coil ring $\tilde{\Omega}_{c_0}$.

From the considerations of Bossavit [2004], Ch. 7, we now briefly want to show how we can derive the density of current originating in an applied voltage. We consider for simplicity that the conductors $\tilde{\Omega}_{c_0}$ consist of a single cylindrical coil ring as in the figure 2.4. Both ends of the ring S^+ and S^- are connected to an electrode, which we assume to be superconducting. The current j flowing through the coil ring is originating from a given potential difference. As in (2.24), we have

$$E = -\frac{\partial A}{\partial t} + \nabla \chi\,, \quad \text{in } \tilde{\Omega}_{c_0}\,, \tag{2.28}$$

with the magnetic potential A and the scalar potential χ. By Ohm's law, we have $j = \mathfrak{s}\, E$, where for simplicity we assume that $\mathfrak{s}$ is a given constant.

Since $\operatorname{div} j = 0$ in the ring, it follows that

$$-\operatorname{div}(\mathfrak{s}\, \nabla \chi) = 0\,, \text{ in } \tilde{\Omega}_{c_0}\,, \tag{2.29}$$

where we have assumed that the magnetic potential A satisfies the Coulomb gauge $\operatorname{div} A = 0$ in the entire domain. On the other hand, we have at the boundary of the electrodes

$$0 = E \times \vec{n} = -\frac{\partial A \times \vec{n}}{\partial t} + \nabla \chi \times \vec{n} = \nabla \chi \times \vec{n}\,, \quad \text{on } S^+,\, S^-\,, \tag{2.30}$$

since $A \times \vec{n}$ vanishes on superconducting surfaces. This means that $\chi = C^+$ at the one electrode, and $\chi = C^-$ at the other one, where $C^{+,-} = C^{+,-}(t)$ depend only on time and are given (applied voltage). At the lateral boundary Γ of the coil, we have $j \cdot \vec{n} = 0$, since no current flows into the vacuum. This means that

$$\frac{\partial \chi}{\partial \vec{n}} = \frac{\partial A \cdot \vec{n}}{\partial t}\,, \quad \text{on } \Gamma\,. \tag{2.31}$$

The relations (2.29), (2.30), (2.31) determine the potential χ uniquely in $\tilde{\Omega}_{c_0}$.

In the first model described in the paragraph (2.2.3), the terms originating in induction A_t are neglected in (2.28), (2.31). The second model corresponds to the case that A_t is neglected only in (2.31). In both cases, an approximation of the potential χ is obtained.

Some further approximations The model described in the paragraph 2.2, and in general the magnetohydrodynamic equations, are difficult to solve numerically in realistic settings. In practice, it is often possible to further reduce the complexity of the model. We want to briefly mention two widely used approximations: the approximation of *low magnetic Reynolds numbers* and the *quasi-stationary approach.*

Low magnetic Reynolds numbers The magnetic Reynolds number Rm, defined in the paragraph B, measures the relative importance of the current due to conduction $\mathfrak{s}\, E$ and of the motion-induced current $\mathfrak{s}\,(v \times B)$ in Ohm's law (3.28). In the case that this number is sufficiently small, (3.28) can often be replaced by $j = \mathfrak{s}\, E$, and Maxwell's equations decouple from the fluid. Consequently, the Lorentz force $j \times B$ in the fluid can be considered as given.

The approximation of low-magnetic Reynolds numbers has been for example investigated mathematically in Peterson [1988]. It is of wide use in Crystal growth with magnetic fields (see Lechner et al. [2007]), its use being mostly justified by dimensional analysis. Precise estimates depending on the other characteristic numbers of the problem are however not available.

Quasi-stationary approach Though Czochralski crystal growth is essentially time-dependent, the long running times (order of days) are a crux for time-dependent simulations. But for the same reason, it is possible to assume that the process is reasonably independent of its beginning phase and of its initial conditions.

Assume that the applied alternating current has a caracteristic frequency $\omega > 0$, which is much higher than the typical relaxation times for momentum and heat-transfer. At the time-scale of the electromagnetic evolution, for example the interval $]t,\, t+2\,\pi/\omega[$, the fluid properties v, p and the temperature θ may be assumed to be stationary. Averaging the equations (2.1) and (2.7) over the interval $]t,\, t + 2\,\pi/\omega[$, one therefore gets

$$\rho_1\,(v\cdot\nabla)v = -\nabla p + \operatorname{div}(2\,\eta\,D\,v) + f(\theta) + [j \times B]_{\mathrm{av}}\,, \tag{2.32}$$

and

$$\rho_1\,c_V\,v\cdot\nabla\theta = \operatorname{div}(\kappa\,\nabla\theta) + 2\,\eta\,D(v,\,v) + [\frac{|j|^2}{\mathfrak{s}}]_{\mathrm{av}}\,, \tag{2.33}$$

where

$$[F]_{\mathrm{av}} := \frac{\omega}{2\,\pi}\int_t^{t+2\,\pi/\omega} F(s)\,ds\,.$$

Sufficiently far from the beginning of the evolution, the electromagnetic fields become independent of their initial conditions, so we may in addition assume that the electromagnetic quantities have reached a *time-harmonic* oscillation regime. This means that we can write

$$j(t,\,x) = \operatorname{Im}(j_0(x)\,\exp(i\,\omega\,t)), \quad H(t,\,x) = \operatorname{Im}(H_0(x)\,\exp(i\,\omega\,t))\,,$$

and so on for the other electromagnetic quantities, where j_0, H_0 are called the amplitudes and are complex valued vector fields, and i is the imaginary unity. One can rewrite the system (2.32) and (2.33) as

$$\begin{aligned} \rho_1\,(v\cdot\nabla)v &= -\nabla p + \operatorname{div}(2\,\eta\,D\,v) + f(\theta) + 1/2\,[\operatorname{Re}(j_0)\times\operatorname{Re}(B_0) + \operatorname{Im}(j_0)\times\operatorname{Im}(B_0)]\,, \\ \rho_1\,c_V\,v\cdot\nabla\theta &= \operatorname{div}(\kappa\,\nabla\theta) + 2\,\eta\,D(v,\,v) + 1/2\,\frac{|\operatorname{Re}(j_0)|^2 + |\operatorname{Im}(j_0)|^2}{\mathfrak{s}}\,. \end{aligned} \tag{2.34}$$

In the time harmonic setting, the relation (2.18) yields

$$\operatorname{curl} E_0 + i\,\omega\,B_0 = 0\,.$$

Solving this equation in connection to (2.34) is therefore sufficient to determine all the unknown quantities of the problem (cp. the model in Rappaz and Touzani [1992]).

Chapter 3

Mathematical setting

3.1 Initial and boundary value problems

In order to formulate the mathematical problem, we at first need to introduce a description of the geometry that fits realistic situations such as represented on Figure 2.1. Note that in most situations it is not realistic to assume that the applied magnetic field is confined to the region of interest for the computation of temperature, the furnace. Therefore, the different phenomena in which we are interested have also to be modeled on different scales[1].

We consider disjoint bounded domains $\tilde{\Omega}_0, \ldots, \tilde{\Omega}_m \subset \mathbb{R}^3$, $(m \geq 1)$, such that the set $\tilde{\Omega}$ defined by $\overline{\tilde{\Omega}} := \bigcup_{i=0}^{m} \overline{\tilde{\Omega}_i}$ is simply connected, and represents the region in which the electromagnetic fields are acting. The domains $\tilde{\Omega}_i$ $(i = 0, \ldots, m)$ represent the different materials filling the region.

We denote by $\Omega \subseteq \tilde{\Omega}$ the bounded domain of interest for temperature distribution. Defining $\Omega_i := \tilde{\Omega}_i \cap \Omega$, we have $\overline{\Omega} := \bigcup_{i=0}^{m} \overline{\Omega_i}$. An example for the region Ω is given in Figure 2.1. We set $\Gamma := \partial\Omega$.

One of the subdomain of Ω is occupied by a liquid. We denote this region by Ω_1.

Further precision is needed. We denote by $\tilde{\Omega}_c \subseteq \tilde{\Omega}$ the region occupied by electrically conducting materials, and by $\tilde{\Omega}_{c_0} \subseteq \tilde{\Omega}_c$ the region where a current source is acting. This region is assumed to be insulated, in the sense that it has a positive distance to the other conductors. It typically consists of coil rings which we can think about as closed. We set $\Omega_c := \tilde{\Omega}_c \cap \Omega$, and define Ω_{c_0} analogously. The set $\tilde{\Omega}_{nc} := \tilde{\Omega} \setminus \tilde{\Omega}_c$ is occupied by electrically nonconducting materials.

One of the different subdomains of the region Ω, say Ω_0, represents an *enclosed cavity* filled with a transparent material. The remaining materials $\Omega_1, \ldots, \Omega_m$ are assumed to be opaque. We define $\overline{\Omega_{\text{op}}} := \bigcup_{i=1}^{m} \overline{\Omega_i}$. The *enclosure property* is satisfied, meaning that

$$\mathbb{R}^3 \setminus \Omega_{\text{op}} \text{ is disconnected.} \tag{3.1}$$

At the boundary of the transparent cavity, heat radiation is emitted, reflected and absorbed. We denote by $\Sigma := \partial\Omega_0$ this boundary.

[1]The geometrical complexity is an essential feature of the problem that we want to study, and deserves to be introduced carefully. The system of equations described in the paragraph 2.2 makes no sense in the standard 'bounded domain Ω in $\mathbb{R}^3$'

To summarize, the geometry contains the following ingredients:

1. A bounded domain $\tilde{\Omega} \subset \mathbb{R}^3$ that represent the region of extension of the electromagnetic fields, and the different materials $\tilde{\Omega}_0, \ldots, \tilde{\Omega}_m$ filling it.

2. A bounded domain $\Omega \subseteq \tilde{\Omega}$ that represents the region of interest for the computation of the temperature, and the different materials $\Omega_0, \ldots, \Omega_m$ filling it.

3. The set Ω_0 is a transparent connected cavity enclosed in Ω (see (3.1)).

4. The set Ω_1 is a vessel filled with the melted semiconducting material.

5. Each material located in Ω is either transparent or opaque. For simplicity, only Ω_0 is assumed to be transparent. The set Ω_{op} of opaque materials consists of the remaining components.

6. Each material located respectively in $\tilde{\Omega}$ and in Ω is either conducting and belongs to $\tilde{\Omega}_c$, resp. to Ω_c, or nonconducting and belongs to $\tilde{\Omega}_{nc}$, resp. to Ω_{nc}.

7. The sets $\tilde{\Omega}_{c_0}$, resp. Ω_{c_0} represent the conductors in which current is applied.

8. Nonlocal radiation interaction take place at the boundary $\partial\Omega_0$ of the transparent cavity. We use the usual notation $\Sigma := \partial\Omega_0$. $\Gamma := \partial\Omega$ is the external boundary of the furnace.

We denote by $T > 0$ a finite time, for instance the end time of the process. We use the notations

$$\tilde{Q} :=]0, T[\times\tilde{\Omega}, \qquad \tilde{Q}_i :=]0, T[\times\tilde{\Omega}_i, \quad \text{for } i = 0, \ldots, m\,.$$

Analogously

$$Q :=]0, T[\times\Omega, \qquad Q_i :=]0, T[\times\Omega_i, \quad \text{for } i = 0, \ldots, m\,,$$

and

$$\tilde{Q}_c :=]0, T[\times\tilde{\Omega}_c, \qquad Q_c :=]0, T[\times\Omega_c \quad \text{etc.}$$

For $t < T$, $\tilde{Q}_t$ denotes the set $]0, t[\times\tilde{\Omega}$, $\tilde{Q}_{i,t}$ denotes the set $]0, t[\times\tilde{\Omega}_i$, etc. We use the notations

$$\mathcal{S} =]0, T[\times\Sigma, \quad \mathcal{C} =]0, T[\times\Gamma\,,$$

and, for $t < T$, $\mathcal{S}_t =]0, t[\times\Sigma$ and $\mathcal{C}_t =]0, t[\times\Gamma$.

3.1.1 The boundary value problem for the stationary system.

A steady-state analysis can be relevant for crystal growth from the melt, in the case that a steady state magnetic field is used to stabilize the melt motion, or in order to describe a quasi-stationary approximation (cp. the paragraph 2.3). Following the paragraph 2.2, we consider the equations

$$\rho_1 \, (v \cdot \nabla) v = -\nabla p + \operatorname{div}(2\,\eta(\theta)\, D\, v) + f(\theta) + j \times B \quad \text{ in } \Omega_1 \,, \tag{3.2}$$

$$\operatorname{div} v = 0 \quad \text{ in } \Omega_1 \,, \tag{3.3}$$

$$f(\theta) = \rho(\theta)\, \vec{g} := \rho_1 \,(1 - \alpha\,(\theta - \theta_1)) \quad \text{ in } \Omega_1 \,, \tag{3.4}$$

$$\rho_1 \, c_V \, v \cdot \nabla\theta = \operatorname{div}(\kappa(\theta)\, \nabla\theta) + 2\,\eta(\theta)\, D(v,v) + \frac{|j|^2}{\mathfrak{s}(\theta)} \quad \text{ in } \Omega \,, \tag{3.5}$$

$$\operatorname{curl} H = j \quad \text{ in } \tilde{\Omega} \,, \tag{3.6}$$

$$\operatorname{curl} E = 0 \quad \text{ in } \tilde{\Omega} \,, \tag{3.7}$$

$$\operatorname{div} B = 0 \quad \text{ in } \tilde{\Omega} \,, \tag{3.8}$$

$$j = \begin{cases} 0 & \text{in } \tilde{\Omega}_{nc} \\ j_0 & \text{in } \tilde{\Omega}_{c_0} \\ \mathfrak{s}(\theta)\,(E + v \times B) & \text{in } \tilde{\Omega}_c \setminus \tilde{\Omega}_{c_0} \end{cases} \,, \tag{3.9}$$

$$\operatorname{div} D = 0 \quad \text{ in } \tilde{\Omega}_{nc} \,, \tag{3.10}$$

$$B = \mu\, H \,, \quad D = \mathfrak{e}\, E \quad \text{ in } \tilde{\Omega} \,, \tag{3.11}$$

with

Unknowns	**Parameters**
v fluid velocity	ρ_1 reference mass density of the fluid
p fluid pressure	η dynamic viscosity
θ absolute temperature	α coefficient of thermal expansion
E electric field strength	θ_1 reference temperature of the fluid
B magnetic induction	c_V specific heat
H magnetic field strength	κ heat conductivity
j electric current density	$\mathfrak{s}$ electrical conductivity
D electric displacement	μ magnetic permeability
	$\mathfrak{e}$ electrical permittivity.

As to the gravitational force $f : \mathbb{R} \longrightarrow \mathbb{R}^3$ given by (3.4), we note that $\rho_1 \, \vec{g} = \nabla \mathcal{G}$ with a scalar potential $\mathcal{G}$. Therefore, we can as well solve the problem with a corrected pressure $\tilde{p} := p + \mathcal{G}$, and the force

$$f(\theta) = -\rho_1 \, \vec{g}\, \alpha\,(\theta - \theta_M) \,, \tag{3.12}$$

where for the reference tempereture θ_1, we have chosen the mean value θ_M of the temperature over the set Ω_1.

We consider the following boundary conditions

$$v = v_g \quad \text{on } \partial\Omega_1\,, \tag{3.13}$$

$$\left[-\kappa(\theta)\,\frac{\partial\theta}{\partial\vec{n}}\right] = R - J \quad \text{on } \Sigma\,, \tag{3.14}$$

$$R = \epsilon\,\sigma\,|\theta|^3\,\theta + (1-\epsilon)\,J\,, \quad \text{on } \Sigma\,, \tag{3.15}$$

$$J = K(R) \quad \text{on } \Sigma\,, \tag{3.16}$$

$$\theta = \theta_g \quad \text{on } \Gamma\,, \tag{3.17}$$

$$\big[H \times \vec{n}\big]_{i,j} = 0\,, \quad \big[B \cdot \vec{n}\big]_{i,j} = 0\,, \quad \big[E \times \vec{n}\big]_{i,j} = 0 \quad \text{on } \partial\tilde{\Omega}_i \cap \partial\tilde{\Omega}_j\,, \tag{3.18}$$

$$B \cdot \vec{n} = 0\,, \quad E \times \vec{n} = 0 \;\; \text{on } \partial\tilde{\Omega}\,. \tag{3.19}$$

As explained in the paragraph 2.2.2, we will rewrite (3.14), (3.15) and (3.16) in the equivalent form

$$\left[-\kappa(\theta)\,\frac{\partial\theta}{\partial\vec{n}}\right] = G(\sigma\,\theta^4) \quad \text{on } \Sigma\,, \tag{3.20}$$

with the linear operator $G := (I - K)\,(I - (1-\epsilon)\,K)^{-1}\,\epsilon$, and the integral operator K introduced in (2.14).

The condition (3.17) does not need further comment. At interfaces between opaque materials, we simply assume the continuity of the conductive heat flux.

The boundary conditions (3.18) are the natural interface conditions for the electromagnetic fields. The conditions (3.19) at the outher boundary model the behavior of the electromagnetic fields at perfectly conducting boundaries. They may be used either to model a magnetic shield, or as an approximation of the condition of vanishing at infinity.

Definition 3.1.1. We will address the problem of finding fields v, H, B, E, D, j and scalars p, θ that satisfy (3.2), (3.3), (3.4), (3.13), (3.5), (3.17), (3.20), (3.6), (3.7), (3.8), (3.9), (3.10), (3.11), (3.18), and (3.19), as Problem (P_{st}).

3.1.2 The initial boundary value problem for the time dependent system

According to the paragraph 2.2, we consider the equations

$$\rho_1\,(\frac{\partial v}{\partial t} + (v\cdot\nabla)v) = -\nabla p + \mathrm{div}(2\,\eta(\theta)\,D\,v) + f(\theta) + j\times B \quad \text{in } Q_1\,, \tag{3.21}$$

$$\mathrm{div}\,v = 0 \quad \text{in } Q_1\,, \tag{3.22}$$

$$f(\theta) = \rho(\theta)\,\vec{g} := -\rho_1\,\alpha\,(\theta-\theta_{\mathrm{M}}) \quad \text{in } Q_1\,, \tag{3.23}$$

$$\rho\,c_V\,(\frac{\partial\theta}{\partial t} + v\cdot\nabla\theta) = \mathrm{div}(\kappa(\theta)\,\nabla\theta) + \{2\,\eta(\theta)\,D(v,v) + \frac{|j|^2}{\mathfrak{s}(\theta)}\}\,\chi_{Q_c\setminus Q_1} \quad \text{in } Q\,, \tag{3.24}$$

$$\mathrm{curl}\,H = j \quad \text{in } \tilde{Q}\,, \tag{3.25}$$

$$\mathrm{curl}\,E + \frac{\partial B}{\partial t} = 0 \quad \text{in } \tilde{Q}\,, \tag{3.26}$$

$$\mathrm{div}\,B = 0 \quad \text{in } \tilde{Q}\,, \tag{3.27}$$

$$j = \mathfrak{s}(\theta)\,\left(-\frac{\partial A}{\partial t} + v\times B\right) + j_g\,, \quad \text{in } \tilde{Q}_c \tag{3.28}$$

$$\mathrm{div}\,D = 0 \quad \text{in } \tilde{Q}_{nc}\,, \tag{3.29}$$

$$B = \mu\,H\,, \quad D = \mathfrak{e}\,E \quad \text{in } \tilde{Q}\,, \tag{3.30}$$

where the same remarks concerning (3.23) are valid as in the paragraph 3.1.1, and where A denotes a magnetic potential for B. We denote by $\chi_{Q_c\setminus Q_1}$ the characteristic function of the set $Q_c\setminus Q_1$. Taking into account the full dissipative heating [2] is mathematically much more difficult in the transient than in the stationary case. In accordance with the full Boussinesq approximation, we therefore consider in (3.24) that the dissipative heating in the fluid is negligible.

We again consider the following boundary conditions

$$v = v_g \quad \text{on }]0,T[\times\partial\Omega_1\,, \tag{3.31}$$

$$\left[-\kappa(\theta)\,\frac{\partial\theta}{\partial\vec{n}}\right] = G(\sigma\,\theta^4) \quad \text{on }]0,T[\times\Sigma\,, \tag{3.32}$$

$$\theta = \theta_g \quad \text{on }]0,T[\times\Gamma\,, \tag{3.33}$$

$$\left[H\times\vec{n}\right]_{i,j} = 0\,, \quad \left[B\cdot\vec{n}\right]_{i,j} = 0\,, \quad \left[E\times\vec{n}\right]_{i,j} = 0 \quad \text{on }]0,T[\times[\partial\tilde{\Omega}_i\cap\partial\tilde{\Omega}_j]\,, \tag{3.34}$$

$$B\cdot\vec{n} = 0\,, \quad E\times\vec{n} = 0 \ \text{ on }]0,T[\times\partial\tilde{\Omega}\,. \tag{3.35}$$

On interfaces between opaque materials, the continuity of the heat flux is assumed. At

[2] The total energy losses result from the Joule effect and from viscous friction.

time zero, we have

$$v(0) = v_0 \quad \text{in } \{0\} \times \Omega_1 \,, \tag{3.36}$$

with the given velocity distribution v_0, and

$$\theta(0) = \theta_0 \quad \text{in } \{0\} \times \Omega \,. \tag{3.37}$$

Finally, we impose the initial condition

$$H(0) = 0 \quad \text{in } \{0\} \times \tilde{\Omega} \,. \tag{3.38}$$

Definition 3.1.2. We define the problem (P) as the problem of finding fields v, H, B, E, D, j and scalars p, θ such that the relations (3.21), (3.22), (3.23), (3.31), (3.36), (3.24), (3.32), (3.33), (3.37), (3.26), (3.25), (3.27), (3.28), (3.29), (3.30), (3.34), and (3.38).

3.1.3 Main assumptions on the data.

The data of the problems consist on the one hand of the geometry, on the other hand of the coefficients ρ, η, α, c_V, κ, ϵ, μ, $\mathfrak{e}$, $\mathfrak{s}$, the boundary data v_g, θ_g and the applied current j_g.

The data are subject to restrictions of *physical nature*. All coefficients are positive. The emissivity on the surface of the bodies attains values in [0, 1]. We impose that $\theta_g \in L^\infty(\mathcal{C})$, and that θ_g is positive everywhere on $\mathcal{C}$. The imposed velocity satisfies $v_g \in [L^\infty(]0,T[\times\partial\Omega_1)]^3$. Since we only consider vessels with fixed walls, we make the additional assumption

$$v_g \cdot \vec{n} = 0 \quad \text{on }]0,T[\times\partial\Omega_1 \,. \tag{3.39}$$

The density of the applied current j_g in the conductor $\tilde{\Omega}_{c_0}$ is given in the form

$$j_g(t,x) = \sin(\omega\, t)\, j_0(x) \quad \text{in }]0,T[\times\tilde{\Omega}_{c_0} \,. \tag{3.40}$$

The parameter $\omega > 0$, the angular frequency of the imposed alternating current, is a given positive constant. The applied power must be finite, that means,

$$j_0 \in [L^2(\tilde{\Omega}_{c_0})]^3 \,. \tag{3.41}$$

In addition, since j_g represents a current, we make the consistency assumption

$$\operatorname{div} j_0 = 0 \,, \text{ in } \tilde{\Omega}_{c_0}, \quad j_0 \cdot \vec{n} = 0 \text{ on } \partial\tilde{\Omega}_{c_0} \,, \tag{3.42}$$

which reflects the conservation of charge in the insulated conductors $\tilde{\Omega}_{c_0}$.

We now impose to the data *mathematical restrictions* valid throughout the thesis. The mass density $\rho := \rho_i$ in Ω_i, $i = 0, \ldots, m$, is a given constant in each material, as well as the specific heat $c_V = c_{Vi}$ and the thermal expansion coefficient α of the fluid.

The coefficients of electrical conductivity, of magnetic permeability, and of heat conductivity are material-dependent. We introduce the abbreviations

$$\mathfrak{s} := \mathfrak{s}_i \,, \quad \mu := \mu_i \,, \quad \kappa := \kappa_i \text{ in each } \tilde{\Omega}_i \text{ for } i = 0, \ldots, m \,. \tag{3.43}$$

Throughout the paper, we assume that there exist positive constants $\mathfrak{s}_l$, $\mathfrak{s}_u$, μ_l, μ_u, κ_l, κ_u, η_l, η_u such that

$$0 < \mathfrak{s}_l \leq \mathfrak{s} \leq \mathfrak{s}_u < +\infty\,, \quad 0 < \mu_l \leq \mu \leq \mu_u < +\infty\,,$$
$$0 < \kappa_l \leq \kappa \leq \kappa_u < +\infty, \quad 0 < \eta_l \leq \eta \leq \eta_u < +\infty\,. \tag{3.44}$$

The emissivity of the surface Σ, denoted by ϵ, is a function of the position. We assume that $\epsilon : \Sigma \longrightarrow \mathbb{R}$ is measurable and that there exists a positive number ϵ_l such that

$$0 < \epsilon_l \leq \epsilon_i \leq 1 \quad \text{on } \partial\Omega_i \cap \Sigma \text{ for } i = 0, \ldots, m\,. \tag{3.45}$$

For the temperature-dependent coefficients, we require that

$$\mathfrak{s}_i,\, \kappa_i,\, \eta \in C\,(\mathbb{R}) \quad \text{for } i = 0, \ldots, m\,. \tag{3.46}$$

For other coefficients, we also require the continuity in each material

$$\mu_i \in C(\overline{\tilde{\Omega}_i}), \quad \epsilon_i \in C(\partial\Omega_i \cap \Sigma)\,. \tag{3.47}$$

For the sake of notational commodity, we introduce the auxiliary function of electric resistivity, that we extend by one to the nonconductors

$$r := \begin{cases} \frac{1}{\mathfrak{s}} & \text{on } \tilde{\Omega}_c \\ 1 & \text{on } \tilde{\Omega}_{nc} \end{cases}, \quad r_l := \mathfrak{s}_u^{-1} \quad r_u := \mathfrak{s}_l^{-1}\,. \tag{3.48}$$

For the geometry, we at this point only want to assume that all domains $\tilde{\Omega}_0, \ldots, \tilde{\Omega}_m$ are of class $\mathcal{C}^{0,1}$. In order to ensure the fundamental properties of Lemma 4.1.2 and Lemma 4.1.5 for the radiation operators, we assume that the surface Σ belongs to $\mathcal{C}^1$ piecewise. In addition, we want to point once again at two main geometrical assumptions: Ω satisfies the enclosure property (3.1), and the source conductors are insulated in the sense that dist $(\tilde{\Omega}_{c_0}, \tilde{\Omega}_c \setminus \tilde{\Omega}_{c_0}) > 0$. In addition, we make the restriction, obvious for the application that we consider, that

$$\operatorname{dist}(\Gamma,\, \Sigma) > 0\,. \tag{3.49}$$

We also need to formulate stronger assumptions with respect to the problem of the higher integrability of the Lorentz force (cp. the section 5). When they are needed, these assumptions will *always be formulated explicitely.* In order to ensure the higher integrability of the Lorentz force, we assume either that

$$\begin{cases} \text{If } \tilde{\Omega}_i,\, \tilde{\Omega}_j \subset \tilde{\Omega}_c, \text{ for some } i,\, j \in \{0, \ldots, m\},\, i \neq j, \text{ then } \operatorname{dist}(\tilde{\Omega}_i,\, \tilde{\Omega}_j) > 0\,. \\ \text{If } \tilde{\Omega}_i \subset \tilde{\Omega}_c, \text{ for some } i \in \{0, \ldots, m\}, \text{ then } \operatorname{dist}(\tilde{\Omega}_i,\, \partial\tilde{\Omega}) > 0\,, \\ \partial\tilde{\Omega}_i \setminus \partial\tilde{\Omega} \in \mathcal{C}^1, \text{ for } i = 0, \ldots, m\,, \quad \partial\tilde{\Omega} \in \mathcal{C}^{0,1}\,. \end{cases} \tag{3.50}$$

or that

$$C\,(1 - \mu_l/\mu_u) < 1\,, \tag{3.51}$$

with the constant C of Lemma 5.2.3, (3), and the constants μ_l, μ_u of (3.44).

3.2 Weak formulation and functional setting

To solve the boundary value problems (P_{st}) and (P) in their original form, the one described in the paragraphs 3.1.1 and 3.1.2, is at the present time too difficult a challenge. Only the initial boundary value problem for the Navier-Stokes equations (3.21), (3.22) in a smooth domain has been defying the mathematical community for two hundred years, and is doing further so. For the study of (P_{st}) and (P), we will therefore employ the tool that has imposed itself as the more powerfull for the analysis of partial differential equations, that is, the approach based on a weak reformulation of the boundary value problems, and on energy estimates, joined to the functional analytic method.

Extensions of the concept of a solution still have, in each particular case, to be considered as problematic. As in the paradigmatic case of the initial boundary value problem for the Navier-Stokes equations, the well-posedness that we would obtain for classical solutions of (P_{st}) and (P) goes lost for the distributional solutions. At the present time, it is not certain whether such limitations are inherent in the functionalanalytic method, or if they come from an incomplete knowledge about the physical phenomena and their mathematical modeling. Still, the functionalanalytic approach remains the foundation of all successful approximation schemes for partial differential equations.

3.2.1 Stationary problems.

Derivation of the integral relations The electromagnetic part of the problem (P_{st}) can be reformulated as a problem involving only the field H of the magnetic field strength. We consider an arbitrary smooth test vector field $\psi \in [C^\infty(\overline{\tilde{\Omega}})]^3$ such that $\operatorname{curl}\psi = 0$ in $\tilde{\Omega} \setminus \Omega_c$. We multiply the relation

$$r(\theta)\ \operatorname{curl} H = E + v \times B \qquad \text{in } \Omega_c\,, \tag{3.52}$$

which follows from (3.9) and (3.6), with $\operatorname{curl}\psi$. For this particular choice of ψ, observe that

$$\int_{\Omega_c} E \cdot \operatorname{curl}\psi = \int_{\tilde{\Omega}} E \cdot \operatorname{curl}\psi = \int_{\tilde{\Omega}} \operatorname{curl} E \cdot \psi + \int_{\partial\tilde{\Omega}} (E \times \psi) \cdot \vec{n} = 0\,,$$

in view of the vector formula (A.14), of (3.7) and of (3.18). Therefore, integration of (3.52) over Ω_c yields the relation

$$\int_{\Omega_c} r(\theta)\ \operatorname{curl} H \cdot \operatorname{curl}\psi = \int_{\Omega_c} (v \times B) \cdot \operatorname{curl}\psi\,.$$

In general, since $\operatorname{curl}\psi = 0$ in $\tilde{\Omega} \setminus \Omega_c$, we write

$$\int_{\tilde{\Omega}} r(\theta)\ \operatorname{curl} H \cdot \operatorname{curl}\psi = \int_{\tilde{\Omega}} (v \times \mu\, H) \cdot \operatorname{curl}\psi\,, \tag{3.53}$$

where we used (3.11) in order to rewrite B as $\mu\, H$. Below, we will introduce a functional setting, in which the last relation (3.53) is sufficient for determining the field H.

In order to derive an integral relation for the velocity v, we as usual multiply the equation (3.2) with a solenoidal vector field $\phi \in [C_c^\infty(\Omega_1)]^3$, and get, after integration over Ω_1

$$\int_{\Omega_1} \rho_1\,(v\cdot\nabla)v\cdot\phi + \int_{\Omega_1} \eta(\theta)\,D(v,\phi) = \int_{\Omega_1} (\operatorname{curl} H \times \mu\,H)\cdot\phi + \int_{\Omega_1} f(\theta)\cdot\phi\,, \qquad (3.54)$$

where we used (3.6) and (3.11) in order to rewrite $j \times B$, and where the bilinear form $D(v,\,\phi)$ is an abbreviation for $Dv\,:\,D\phi$ (cp. (2.2)).

Finally, to derive an integral relation for the temperature, we multiply the relation (3.5) by a function $\xi \in C^\infty(\overline{\Omega})$ that vanishes on Γ, and obtain the relation

$$\int_\Omega \rho_1\,c_V\,v\cdot\nabla\theta\,\xi + \int_\Omega \kappa(\theta)\,\nabla\theta\cdot\nabla\xi + \int_\Sigma G(\sigma\,|\theta|^3\,\theta)\,\xi$$

$$= \int_\Omega r(\theta)\,|\operatorname{curl} H|^2\,\xi + \int_{\Omega_1} \eta(\theta)\,D(v,v)\,\xi\,, \qquad (3.55)$$

where we used (3.6) in order to eliminate j, and the boundary condition (3.20).

Functional spaces We need to find functional spaces in which the left-hand sides of the relations (3.53), (3.54) and (3.55) can be interpreted as coercive duality products.

For the electromagnetic part of the problem, spaces of vector fields with generalized *curl* and *div* are needed. We first introduce

$$L^2_{\text{curl}}(\tilde\Omega) \;:=\; \left\{ H \in [L^2(\tilde\Omega)]^3 \;\middle|\; \operatorname{curl} H \in [L^2(\tilde\Omega)]^3 \right\},$$

where the differential operator *curl* is intended in its generalized sense (some properties of the generalized *curl* operator are recalled in section 5.1 below). As it is well known, $L^2_{\text{curl}}(\tilde\Omega)$ is a Hilbert space with respect to the product

$$(H_1\,,\,H_2)_{L^2_{\text{curl}}(\tilde\Omega)} := \int_{\tilde\Omega} (\operatorname{curl} H_1\cdot\operatorname{curl} H_2 + H_1\cdot H_2)\;.$$

A natural context in which to search for the field H is the space

$$\mathcal{H}(\tilde\Omega) \;:=\; \left\{ H \in\; L^2_{\text{curl}}(\tilde\Omega) \middle|\; \operatorname{curl} H = 0 \text{ in } \tilde\Omega\setminus\tilde\Omega_c \right\}. \qquad (3.56)$$

Obviously, this is a closed linear subspace of $L^2_{\text{curl}}(\tilde\Omega)$. In order to account for the fact that the current is prescribed in $\tilde\Omega_{c_0}$, we will also need the space

$$\mathcal{H}^0(\tilde\Omega) \;:=\; \left\{ H \in\; L^2_{\text{curl}}(\tilde\Omega) \middle|\; \operatorname{curl} H = 0 \text{ in } \tilde\Omega\setminus\Omega_c \right\}. \qquad (3.57)$$

If μ is given by (3.43) and satisfies (3.44), it is possible to deal with the divergence constraint (3.8) and the boundary conditions (3.18) by introducing

$$\mathcal{H}_\mu(\tilde\Omega) := \left\{ H \in \mathcal{H}(\tilde\Omega) \;\middle|\; \operatorname{div}(\mu\,H) = 0 \text{ in } \tilde\Omega\,;\; \mu\,H\cdot\vec{n} = 0 \text{ on } \partial\tilde\Omega \right\}, \qquad (3.58)$$

and, correspondingly,

$$\mathcal{H}^0_\mu(\tilde\Omega) := \Big\{ H \in \mathcal{H}^0(\tilde\Omega) \,\Big|\, \operatorname{div}(\mu\, H) = 0 \text{ in } \tilde\Omega\,;\ \mu\, H \cdot \vec{n} = 0 \text{ on } \partial\tilde\Omega \Big\}\,. \tag{3.59}$$

Here again, the divergence constraint and the boundary condition are intended in the generalized sense.

The appropriate setting for the Navier-Stokes equations is widely known. We need the spaces

$$D^{1,2}(\Omega_1) := \Big\{ u \in [W^{1,2}(\Omega_1)]^3 \,\Big|\, \operatorname{div} u = 0 \text{ almost everywhere in } \Omega_1 \Big\}\,,$$

$$D^{1,2}_0(\Omega_1) := \Big\{ u \in [W^{1,2}_0(\Omega_1)]^3 \,\Big|\, \operatorname{div} u = 0 \text{ almost everywhere in } \Omega_1 \Big\}\,. \tag{3.60}$$

For the mathematical setting of the stationary heat equation with radiation boundary conditions, we need spaces of functions whose trace is integrable to a higher exponent than the one given by Sobolev's embedding relations. These are the spaces

$$V^{p,q}(\Omega) := \Big\{ \theta \in W^{1,p}(\Omega) \,\Big|\, \gamma(\theta) \in L^q(\Sigma) \Big\}\,, \quad 1 \le p \le \infty, \quad 4 \le q \le \infty\,, \tag{3.61}$$

where γ denotes the trace operator. The subscript Γ will indicate the subspace consisting of all functions whose trace vanishes on the boundary part Γ. The norm in $V^{p,q}(\Omega)$ is of course given by $\|\cdot\|_{W^{1,p}(\Omega)} + \|\gamma(\cdot)\|_{L^q(\Sigma)}$.

We give the following definition, which already anticipates on our results, at this stage without comment.

Definition 3.2.1. Let the assumptions formulated in the paragraph 3.1.3 be satisfied, with the data v_g, θ_g being stationary and $j_g = j_0$. Let one of the additional assumptions (3.50), (3.51) be valid. We call *weak solution* to (P_{st}) a triple

$$\{v,\, H,\, \theta\} \in D^{1,2}(\Omega_1) \times \mathcal{H}_\mu(\tilde\Omega) \times \bigcap_{1\le p<3/2} V^{p,4}(\Omega)\,,$$

such that $v = v_g$ on $\partial\Omega_1$, $\theta = \theta_g$ on Γ, $\operatorname{curl} H = j_0$ in $\tilde\Omega_{c_0}$ and the integral relations

$$\int_{\Omega_1} \rho_1\,(v\cdot\nabla)v\cdot\phi + \int_{\Omega_1} \eta(\theta)\, D(v\,,\,\phi) = \int_{\Omega_1} (\operatorname{curl} H \times \mu\, H)\cdot\phi + \int_{\Omega_1} f(\theta)\cdot\phi\,, \tag{3.62}$$

$$\int_{\tilde\Omega} r(\theta)\, \operatorname{curl} H \cdot \operatorname{curl}\psi = \int_{\Omega_1} \big(v \times \mu\, H\big)\cdot \operatorname{curl}\psi\,, \tag{3.63}$$

$$\int_{\Omega_1} \rho_1\, c_V\, v\cdot\nabla\theta\,\xi + \int_\Omega \kappa(\theta)\nabla\theta\cdot\nabla\xi + \int_\Sigma G(\sigma\,\theta^4)\,\xi$$
$$= \int_\Omega \Big(r(\theta)\,|\operatorname{curl} H|^2 + \eta(\theta)\, D(v\,,\,v)\,\chi_{\Omega_1}\Big)\,\xi\,, \tag{3.64}$$

are satisfied for all $\{\phi,\, \psi,\, \xi\} \in D^{1,2}_0(\Omega) \times \mathcal{H}^0_\mu(\tilde\Omega) \times W^{1,r}_\Gamma(\Omega)$ with $r > 3$.

Remark 3.2.2 (Well-posedness of Definition 3.2.1)**.** The assumption (3.50) or (3.51) ensures, in view of Lemma 5.2.3, (2) or (3), that

$$\operatorname{curl} H \times \mu H \in [L^{6/5}(\tilde{\Omega})]^3 . \tag{3.65}$$

On the other hand, the assumption (3.45), together with the regularity $\Sigma \in \mathcal{C}^1$ piecewise, ensures that the definition (2.16) of the radiation operator G is well posed, and that G is continuous form $L^1(\Sigma)$ into itself (cp. Lemma 4.1.5, (1) and (4)).

The well-posedness of definition 3.2.1 is therefore readily checked.

3.2.2 Transient problems.

Derivation of the integral relations The derivation of a weak formulation is quite similar to $(P_{\rm st})$. However, because of the electromagnetic law of induction, the current cannot be considered as imposed in $\tilde{\Omega}_{c_0}$.

Therefore, we here take a smooth test vector field ψ such that $\operatorname{curl}\psi = 0$ in the nonconductors $\tilde{\Omega}_{nc}$. We multiply the relation (2.26) by $\operatorname{curl}\psi$, and integrate over $\tilde{\Omega}_c$. Note that the boundary condition $B \cdot \vec{n} = 0$ on $]0,T[\times\partial\tilde{\Omega}$ implies for the magnetic potential that $A \times \vec{n} = 0$ on $]0,T[\times\partial\tilde{\Omega}$. We then observe that

$$\int_{\tilde{\Omega}_c} -\frac{\partial A}{\partial t} \cdot \operatorname{curl}\psi = \int_{\tilde{\Omega}} -\frac{\partial}{\partial t} \operatorname{curl} A \cdot \psi = \int_{\tilde{\Omega}} -\frac{\partial B}{\partial t} \cdot \psi .$$

Therefore, we obtain the relation

$$\int_{\tilde{\Omega}} \frac{\partial B}{\partial t} \cdot \psi + \int_{\tilde{\Omega}} r(\theta) \operatorname{curl} H \cdot \operatorname{curl}\psi = \int_{\tilde{\Omega}} (v \times B) \cdot \operatorname{curl}\psi + \int_{\tilde{\Omega}} r(\theta)\, j_g \cdot \operatorname{curl}\psi .$$

Note that $v \neq 0$ only in Ω_1. We use the relation (3.30), and the fact that μ depends only on the position in oder to obtain, after integration over $[0,T]$,

$$\int_{\tilde{Q}} \mu \frac{\partial H}{\partial t} \cdot \psi + \int_{\tilde{Q}} r(\theta) \operatorname{curl} H \cdot \operatorname{curl}\psi = \int_{\tilde{\Omega}} (v \times \mu H) \cdot \operatorname{curl}\psi + \int_{\tilde{\Omega}} r(\theta)\, j_g \cdot \operatorname{curl}\psi . \tag{3.66}$$

As to the Navier-Stokes equations, we easily derive the relation

$$\int_{Q_1} \rho_1 \frac{\partial v}{\partial t} \cdot \phi + \int_{Q_1} \rho_1 (v \cdot \nabla) v \cdot \phi + \int_{Q_1} \eta(\theta)\, D(v\,,\,\phi)$$
$$= \int_{Q_1} (\operatorname{curl} H \times \mu H) \cdot \phi + \int_{Q_1} f(\theta) \cdot \phi . \tag{3.67}$$

Using the same tools as in the steady state case, the weak formulation of (3.24) is

$$\int_Q \rho\, c_V \frac{\partial \theta}{\partial t} \xi + \int_Q \rho_1\, c_{V1}\, v \cdot \nabla\theta\, \xi + \int_Q \kappa(\theta) \nabla\theta \cdot \nabla\xi + \int_{\mathbb{S}} G(\sigma\,\theta^4)\, \xi$$
$$= \int_{Q_c \setminus Q_1} r(\theta)\, |\operatorname{curl} H|^2\, \xi . \tag{3.68}$$

Mathematical setting In the context of the time-dependent problem, we use the evolution spaces $L^p(0,T;X)$, where X is a given Banach space. In addition, we use for $1 \le p,\, q < \infty$ the notation

$$L^{p,q}(Q) \,:=\, \left\{ u \in L^1(Q) \,\middle|\, \int_0^T \left(\int_\Omega |u|^q \, dx \right)^{\frac{p}{q}} \, dt \, < \infty \right\} ,$$

and for $p = \infty$,

$$L^{\infty,q}(Q) \,:=\, \left\{ u \in L^1(Q) \,\middle|\, \operatorname*{ess\,sup}_{t \in]0,T[} \left(\int_\Omega |u(t,\, x)|^q \, dx \right)^{\frac{1}{q}} \, < \infty \right\} .$$

Analogously, one can define the spaces $L^{p,q}(\mathcal{S})$. We use also the notations $L^p(Q)$, $L^p(\mathcal{S})$ instead of $L^{p,p}(Q)$, $L^{p,p}(\mathcal{S})$. For $1 \le p < \infty$, we use the spaces

$$W_p^{1,0}(Q) := \left\{ u \in L^p(Q) \,\middle|\, \exists\, u_{x_i} \in L^p(Q) \ (i = 1, \ldots, 3) \right\} ,$$

and

$$W_p^1(Q) := \left\{ u \in W_p^{1,0}(Q) \,\middle|\, \exists\, u_t \in L^p(Q) \right\} ,$$

where all partial derivatives are intended in the weak sense. The space $V_2^{1,0}(Q)$ (resp. $V_2^1(Q)$) consists of all $u \in W_2^{1,0}(Q)$ (resp. $u \in W_2^1(Q)$) such that

$$\operatorname*{ess\,sup}_{t \in]0,T[} \int_\Omega u^2(t,x) \, dx < \infty \, .$$

Referring to (3.58), we introduce the space $\mathcal{H}_\mu(\tilde{Q})$, etc. We set $D_2^{1,0}(Q_1)$ as the subspace of $W_2^{1,0}(Q_1)$ which contains all elements with almost everywhere vanishing divergence.

We will use the set

$$\mathcal{D}(\Omega_1) := \left\{ u \in [C_c^\infty(\Omega_1)]^3 \,\middle|\, \operatorname{div} u = 0 \text{ in } \Omega_1 \right\} , \tag{3.69}$$

and the set

$$\mathfrak{H}(\tilde{\Omega}) := \left\{ u \in [C^\infty(\overline{\tilde{\Omega}})]^3 \,\middle|\, \operatorname{curl} u = 0 \text{ in } \tilde{\Omega}_{nc} \right\} . \tag{3.70}$$

Observe that we do not introduce time-dependent analogs for the spaces $V^{p,q}(\Omega)$ given by (3.61). The reason is that will not need such spaces, since the properties of the operator G do not allow to prove coercivity results on similar spaces of functions depending on x and t. A straightforward definition of a weak solution is the following:

Definition 3.2.3. Let the assumptions formulated in the paragraph 3.1.3 be satisfied, without the additional assumptions (3.50) and (3.51). We call *weak solution* to (P) a triple

$$\{v,\, H,\, \theta\} \in D_2^{1,0}(Q_1) \times \mathcal{H}_\mu(\tilde{Q}) \times \bigcap_{1 \le p < 5/4} W_p^{1,0}(Q) \cap L^4(\mathcal{S}) \, ,$$

such that $v = v_g$ on $]0,T[\times\partial\Omega_1$, $\theta = \theta_g$ on $]0,T[\times\Gamma$, and the integral relations

$$-\int_{Q_1} \rho_1\, v\cdot\frac{\partial\phi}{\partial t} + \int_{Q_1} \rho_1\,(v\cdot\nabla)v\cdot\phi + \int_{Q_1} \eta(\theta)\, D(v\,,\,\phi)$$

$$= \int_{\Omega_1} v_0\cdot\phi(0) + \int_{Q_1} (\operatorname{curl} H\times\mu\, H)\cdot\phi + \int_{Q_1} f(\theta)\cdot\phi\,, \tag{3.71}$$

$$-\int_{\tilde Q} \mu\, H\cdot\frac{\partial\psi}{\partial t} + \int_{\tilde Q} r(\theta)\ \operatorname{curl} H\cdot\operatorname{curl}\psi = \int_{Q_1} \big(v\times\mu\, H\big)\cdot\operatorname{curl}\psi\ + \int_{\tilde Q} r(\theta)\, j_g\cdot\operatorname{curl}\psi, \tag{3.72}$$

$$-\int_{Q} \rho\, c_V\,\theta\,\frac{\partial\xi}{\partial t} - \int_{Q_1} \rho_1\, c_V\,\theta\, v\cdot\nabla\xi + \int_{Q} \kappa(\theta)\nabla\theta\cdot\nabla\xi + \int_{\mathcal{S}} G(\sigma\,\theta^4)\,\xi$$

$$= \int_{\Omega} \theta_0\,\xi(0) + \int_{Q\setminus Q_1} r(\theta)\,|\operatorname{curl} H|^2\,\xi\,, \tag{3.73}$$

are satisfied for all $\{\phi,\,\psi,\,\xi\}\in C^\infty(0,T;\mathcal{D}(\Omega_1))\times C^\infty(0,T;\mathfrak{H}(\tilde\Omega))\times C^\infty(\overline{Q})$, such that $\phi(T) = 0 = \psi(T)$ and such that $\xi = 0$ in $\{T\}\times\Omega$ and on $\mathcal{C}$.

For the time-dependent problem, we have to face the difficulty that the radiation operators exert no regularization in the time-variable, even for smooth surfaces Σ. For this reason, we are not able to use the term $G(\sigma\,\theta^4)$ to obtain a global estimate on the total emitted radiation $\int_0^T\int_\Sigma \theta^4$ as for the stationary problem.

In view of difficulties inherent in the mathematical theory of the Navier-Stokes equations, we are only able to prove, for the full problem, an energy inequality.

Therefore, we have to introduce also a weaker definition.

Definition 3.2.4 (Weak solution with defect measure)**.** Let a triple

$$\{v,\,H,\,\theta\}\in D^{1,2}(Q_1)\times\mathcal{H}_\mu(\tilde Q)\times\bigcap_{1\le p<5/4} W^{1,0}_p(Q)\,,$$

satisfy $v = v_g$ on $]0,T[\times\partial\Omega_1$, $\theta = \theta_g$ on $]0,T[\times\Gamma$, as well as the integral relations (3.71) and (3.72) for all $\{\phi,\,\psi\}\in C^\infty(0,T;\mathcal{D}(\Omega_1))\times C^\infty(0,T;\mathfrak{H}(\tilde\Omega))$ such that $\phi(T) = 0 = \psi(T)$. Assume that we can find an operator $\mathcal{G}:\bigcap_{1\le p<5/4} W^{1,0}_p(Q)\to[W^1_{q,\mathcal{C}}(Q)]^*$ $(q>5)$, such that $\langle\mathcal{G}(\theta),\,1\rangle = 0$ and such that

$$-\langle\mathcal{G}(\theta),\,\xi\rangle + \int_{\mathcal{S}} \sigma\,\theta^4\,\xi \ge \int_{\mathcal{S}} \sigma\,\theta^4\,\mathbf{H}(\xi)\,,$$

for all $\xi\in W^1_{q,\mathcal{C}}(Q)$, $\xi\ge 0$ almost everywhere in Q. Assume that there exists a Borel regular, positive Radon measure $\nu\in\mathcal{M}(\overline{Q})$ such that $\{v,\,H,\,\theta\}$ satisfies the integral relation

$$-\int_{Q} \rho\, c_V\,\theta\,\frac{\partial\xi}{\partial t} - \int_{Q_1} \rho_1\, c_V\,\theta\, v\cdot\nabla\,\xi + \int_{Q} \kappa(\theta)\nabla\theta\cdot\nabla\xi + \langle\mathcal{G}(\theta),\,\xi\rangle$$

$$= \int_{\Omega} \theta_0\,\xi(0) + \int_{Q\setminus Q_1} r(\theta)\,|\operatorname{curl} H|^2\,\xi + \int_{[0,T]\times\partial\Omega_c} \xi\, d\,\nu\,, \tag{3.74}$$

for all $\xi \in W^1_{q,\mathcal{C}}(Q)$, such that $\xi = 0$ in $\{T\} \times \Omega$.

Assume that there further exists a Borel regular, signed Radon measure $\tilde{\nu} \in \mathcal{M}(\overline{Q})$ such that $\tilde{\nu} \leq \nu$, and

$$- \int_Q \rho\, c_V\, \theta\, \frac{\partial \xi}{\partial t} - \int_{Q_1} \rho_1\, c_V\, \theta\, v \cdot \nabla\, \xi + \int_Q \kappa(\theta) \nabla\theta \cdot \nabla\xi + \int_{\mathcal{S}} \sigma\, \theta^4\, G(\xi)$$
$$= \int_\Omega \theta_0\, \xi(0) + \int_{Q \setminus Q_1} r(\theta)\, |\operatorname{curl} H|^2\, \xi + \int_{[0,T] \times \partial\Omega_c} \xi\, d\,(\nu - \tilde{\nu})\,, \tag{3.75}$$

for all $\xi \in C^\infty(\overline{Q})$, such that $\xi = 0$ in $\{T\} \times \Omega$ and on $\mathcal{C}$, and such that

$$\int_0^T \max_{\overline{\Omega}} |\xi(t)| \int_\Sigma \theta^4(t)\, dt\ < \infty\,. \tag{3.76}$$

Then we call $\{v,\, H,\, \theta\}$ a *weak solution* of (P) with *defect measure.*

Remark 3.2.5. The defect measure has two parts. The part ν is due to the concentration of the heat sources in the boundary of the electrical conductors. The part $\tilde{\nu}$ is due to nonlocal radiation and remains concentrated in the boundary of the transparent cavity.

The introduction of a generalization $\mathcal{G}$ of the operator G is motivated by the fact that the boundary condition (2.9) can make sense independently of the integrability of θ^4 on the surface $\mathcal{S}$. The second integral relation (3.75) can then be interpreted as follows. For test functions with (3.76), we can prove that the integral $\int_{\mathcal{C}} \theta^4\, G(\xi)$ makes sense, and that the representation

$$\langle \mathcal{G}(\theta),\, \xi \rangle = \int_{\mathcal{C}} \sigma\, \theta^4\, G(\xi) + \tilde{\nu}(\xi)\,, \tag{3.77}$$

is valid.

3.3 State of the research and main results

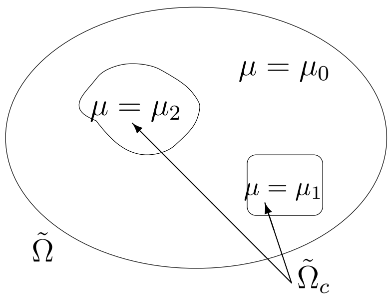
Stationary problems It was pointed out at the beginning of the introduction that the problems (P_{st}) and (P) described in the paragraph 3.1.1 and 3.1.2 have never been investigated. However, there is a non negligible amount of literature that gives hints about how problems such as temperature-dependent coefficients, right-hand sides L^1, transmission conditions have to be treated mathematically. In this paragraph, we want to give a short survey of this literature, restricting ourselves to cite the results more directly related.

The boundary value problem for system of *stationary MHD* (see (1.1), (1.2), (1.3), (1.4), (1.5), (1.6), (1.7a), (1.7b)) corresponds to the problem (P_{st}) in an isothermal setting. The coefficients η, $\mathfrak{s}$ are given functions of the position, as well as the external force term f in the right-hand side of the Navier-Stokes equations. In comparison to the Navier-Stokes equations for a viscous incompressible fluid, the presence of the term $\operatorname{curl} H \times \mu\, H$ represents an additional difficulty for the analysis. As a matter of fact, for general vector fields $H \in L^2_{\text{curl}}(\tilde{\Omega})$, we have only $\operatorname{curl} H \times \mu\, H \in [L^1(\tilde{\Omega})]^3$.

It has been shown that this difficulty can be overcome under suitable geometrical restrictions. In the papers Ladyzhenskaja and Solonnikov [1960], Ladyzhenskaja and Solonnikov [1977], it is proved that for a domain of the form $\overline{\tilde{\Omega}} = \bigcup_{i=0}^{m} \overline{\tilde{\Omega}_i}$, one has

$$\mathcal{H}_\mu(\tilde{\Omega}) \hookrightarrow \bigcap_{i=0}^{m} [W^{1,2}(\tilde{\Omega}_i)]^3 \,, \tag{3.78}$$

with continuous injection, provided that the magnetic permeability μ is piecewise smooth, and that the jump interfaces $\partial\tilde{\Omega}_i \cap \partial\tilde{\Omega}_j$ as well as the outer boundary $\partial\tilde{\Omega}$ belong to the class $\mathcal{C}^2$. In addition, the heterogeneous conductors are not allowed to be in contact with each other (see the Figure 3.3, and compare with the condition 3.50). Note that (3.78) implies, in view of Sobolev's inequality, that $\operatorname{curl} H \times \mu\, H \in [L^{3/2}(\tilde{\Omega})]^3$ whenever $H \in \mathcal{H}_\mu(\tilde{\Omega})$. Less general results were used for example in Duvaut and Lions [1972], Sermange and Temam [1983], Gunzburger et al. [1991]: in these papers a smooth magnetic permeability was assumed, so that no transmission conditions for the magnetic field arise. The more recent studies Meir and Schmidt [1996], Meir and Schmidt [1999] devoted to the stationary MHD equations allow for a nonhomogeneous magnetic permeability, but they are also based on the relatively old result (3.78).



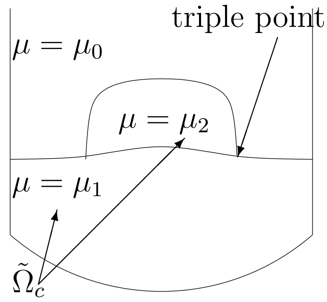


Figure 3.1: Left-hand side: the situation considered in the paper Ladyzhenskaja and Solonnikov [1960] with isolated conductors in the vacuum. Right-hand side: the presence of triple points is not compatible with the assumption of $\mathcal{C}^2$ interfaces.

On the other hand, the investigation of stationary coupled problems involving either the *Navier-Stokes system*, or *Maxwell's system*, and the *heat equation*, can be handled analytically in three dimensions without regularity theory thanks to fundamental results concerning nonlinear elliptic problems with fixed right-hand side in L^1. In papers such as Rakotoson [1991], Boccardo and Gallouët [1989], Boccardo and Gallouët [1992a], Boccardo and Gallouët [1992b], techniques are developed to obtain a-priori estimates for the temperature gradient in the norm of $L^p(\Omega)$ for $1 \leq p < n/(n-1)$ arbitrary, n being the space dimension. These estimates are also sufficient to prove the existence of stationary weak solutions for problems such as the stationary motion of heat conducting incompressible viscous fluids, or resistance heating problems, shortly for problems coupled by temperature-dependent coefficients and by quadratic heat sources of the form (2.8).

An important specificity of the problems that we want to study in this thesis is the additional nonlocal and nonlinear term $G(\theta^4)$, originating from the *radiation boundary conditions*. Existence results concerning the stationary heat equation with this type of boundary condition in an enclosed cavity were for the first time proved in Laitinen and

Tiihonen [2001], under the restriction that the boundary Σ of the cavity must have the smoothness $C^{1,\alpha}$, with $\alpha > 0$. Only right-hand sides in $L^{6/5}(\Omega)$ can be handled with this theory. Therefore, existence and uniqueness of weak solutions to the heat equation with nonlocal radiation boundary conditions is a topic never studied for piecewise smooth boundaries, and for lower integrable right-hand sides.

Finally, a last specificity of the problem $(P_{\rm st})$ consists in the temperature-dependence of the gravity force term $\rho(\theta)\,\vec{g}$ in the Navier-Stokes equations. As underlined in the paragraph 2.3, the Boussinesq model disturbs mass and global energy balance, so that a global energy estimate cannot be proved without additional assumptions. Fortunately, we will see that the needed assumption, the smallness of the coefficient of thermal expansion, is consistent with the conditions of derivation of the Boussinesq approximation. In Lukaszewicz [1988], the function ρ was simply truncated.

Transient problems The existence of weak solutions to the *time-dependent MHD equations* has been proved in the papers Ladyzhenskaja and Solonnikov [1960], Ladyzhenskaja and Solonnikov [1977], under the same kind of geometrical restrictions as in the stationary case. Uniqueness is obtained only for strong solutions. Note that we here speak about solutions that are 'weak' in the sense of the mathematical theory of the Navier-Stokes equations[3] and are not known to satisfy for $t \in]0, T]$ the energy identity

$$\int_{\Omega_1} \frac{\rho}{2}\,|v(t)|^2 + \int_{Q_{t,1}} \eta\, D(v,\, v) = \int_{\Omega_1} \frac{\rho}{2}\,|v_0|^2 + \int_{Q_{t,1}} (\operatorname{curl} H \times \mu\, H)\cdot v + \int_{Q_{t,1}} f\cdot v\,. \tag{3.79}$$

The magnetic field strength H is, as well, not known to satisfy the energy equality

$$\begin{aligned}\int_{\tilde\Omega} \frac{\mu}{2}\,|H(t)|^2 &+ \int_{\tilde Q_t} r\,|\operatorname{curl} H|^2 \\ &= \int_{\tilde\Omega} \frac{\mu}{2}\,|H_0|^2 + \int_{\tilde Q_t} (v\times \mu\, H)\cdot \operatorname{curl} H + \int_{\tilde Q_{c_0,t}} r\, j_g \cdot \operatorname{curl} H\,.\end{aligned} \tag{3.80}$$

This lack of knowledge concerning the energetic behavior of weak solutions leads to difficulties when the system is coupled to energy balance. Though similar a-priori estimates as in the elliptic case are available for problems with fixed right-hand sides in L^1, typical approximation schemes for the time-dependent coupled problems lead only to an energy *inequality*, resp. to supsolutions, of the type (cp. Naumann [2006])

$$\rho_i\, c_{Vi} \frac{\partial\theta}{\partial t} - \operatorname{div}(\kappa(\theta)\,\nabla\theta) \geq \frac{|j|^2}{\mathfrak{s}(\theta)}\,. \tag{3.81}$$

Therefore, *defect measures* appear in the solution, meaning that the relation

$$\rho_i\, c_{Vi} \frac{\partial\theta}{\partial t} - \operatorname{div}(\kappa(\theta)\,\nabla\theta) = \frac{|j|^2}{\mathfrak{s}(\theta)} + \nu\,, \tag{3.82}$$

[3] A 'weak' solution, that is a distributional solution that satisfies the integral relation (3.71), is called 'strong' under an additional regularity condition, such as $v \in [L^{\infty,4}(Q_1)]^3$

is satisfied in the distributional sense with a positive regular measure ν. The concept of a solution with defect measure is consistent in the sense that the measure vanishes for weak solutions that satisfy the energy equality. A similar defect arises in the study of coupled problems for compressible fluids, but has a more elegant solution exposed for instance in Ducomet and Feireisl [2006]. An interesting and difficult problem consist in studying the properties of the measure ν, for example its concentration behavior. This is a widely open field.

Concerning the additional nonlocal and nonlinear radiation term $G(\theta^4)$ in (P), the same remarks are valid as in the stationary case. An existence result was stated in Metzger [1999] for non enclosed cavities. The existence result of Laitinen and Tiihonen [2001] claims validity for enclosures provided that $\Sigma \in C^{1,\alpha}$, with $\alpha > 0$, and that the heat-sources are of class $L^2(Q)$[4]. No results are known on the heat equation with nonlocal radiation boundary conditions in situations such as piecewise smooth boundaries or for lower integrable right-hand sides.

The temperature-dependent gravity force term $f(\theta)$ in the Navier-Stokes equations constitutes an additional source of difficulties, and cannot be controlled without additional assumptions on the data.

To summarize, we can mention four facts that make the problems (P_{st}) and (P) analytically difficult to handle:

(1) No obvious regularity argument for the system of MHD allows to prove that he right-hand side of the heat equation is in a better space than $L^1(\Omega)$, resp. $L^1(Q)$.

(2) There is no theory available for handling the nonlocal radiation boundary operator G in relation with a L^1 right-hand side.

(3) In the generalized setting of the Maxwell system, the Lorentz force $\operatorname{curl} H \times \mu H$ as right-hand side of the Navier-Stokes equations belongs *a priori* only to $[L^1(\Omega_1)]^3$, resp. $[L^1(Q_1)]^3$.

(4) The temperature-dependent buoyancy forces introduced by the Boussinesq model disturb the global energy balance.

Main results We would like to investigate the well-posedness of the problems (P), (P_{st}) allowing for *piecewise smooth boundaries*. Polyhedral boundaries arise naturally in the modeling of apparatus such as in Figure 2.1, as well as in the analysis of most discretization schemes. In addition, one must not forget that the interfaces crystal-melt and melt-crystal are in reality free surfaces, which increases the reluctance to work with strong regularity assumptions on the domain regularity (c.p. Figure 3.3).

We are able to prove *existence of solutions* to the problem (P) in this very general setting, if we weaken the concept of a solution as in Definition 3.2.4. We first obtain a result by truncating the buoyancy forces (2.5) in the section 7.1. We obtain the existence result for the full problem (see the section 7.2) under the assumption that the density variations $\alpha\,(\theta - \theta_{\text{Ref}})$ (c. p. (2.5)) remain sufficiently small. The last assumption is realistic, since it is a condition of validity of the Boussinesq model.

[4]The proof of these results cannot however be regarded as complete

We also obtain some informations about the *concentration behavior of the defect measure*. The defect measure comes for one part from the concentration of the heat sources. Since the dissipative heating is neglected in the fluid, the measure is concentrated in the boundary of the electrical conductors and reflects typical physical phenomena such as the skin effect. For the other part, the measure is originated in the lack of an estimate for the radiation energy in the transparent cavity, and concentrates in subset of the boundary Σ with zero surface measure.

The existence of a gap between the classes of functions in which it is possible to prove existence and to prove uniqueness is a recurrent feature in the study of nonlinear parabolic problems, especially the Navier-Stokes system. We are able to prove the uniqueness of solutions that satisfy

$$\operatorname*{ess\,sup}_{t\in[0,T]} \left\{ \|v(t)\|_{[L^4(Q_1)]^3} + \|\operatorname{curl} H(t)\|_{[L^3(\tilde{Q})]^3} + \|\theta(t)\|_{L^4(\Sigma)} \right\} < \infty\,,$$

but not their existence (see the section 7.3).

We could present the same existence theory in the case of the stationary problem (P_{st}) without the occurrence of a defect measure. However, we can in this case formulate precise assumptions that lead to the existence of stronger solutions (cp. the definition 3.2.1). On the one hand, we must assume that the boundary Σ of the transparent cavity belongs to $\mathcal{C}^{1,\alpha}$ for some $\alpha > 0$. This allows to prove regularizing properties of the radiation operators K, G (see the sections 4.2 and 4.3) that lead to an estimate of the total emitted radiation $\int_\Sigma \theta^4$, and allow to get rid of one part of the measure.

On the other hand, we can make different assumptions on the regularity of the magnetic permeability μ and on its jump interfaces that lead to the higher-integrability of the Lorentz force required to obtain (3.79) and (3.80) in the stationary context (see the section 5). These assumptions are:

1. Either the interfaces are of class $\mathcal{C}^1$ and the permeability is uniformly continuous in each subdomain $\tilde{\Omega}_i$ (cp. (3.50));

2. Or the jumps of the magnetic permeability remain 'small' compared to a certain constant that depends only on the domain $\tilde{\Omega}$ (cp. (3.51)).

Existence is obtained in the section 6.1 under the assumption of the global boundedness of the buoyancy forces. Under the same kind of smallness assumption as above, we obtain existence for the genuine Boussinesq model in the section (6.2). In both sections, we need as in other publications the smallness of the velocity imposed at the boundary of the fluid.

Again, the question of the uniqueness of weak solutions is a critical issue, and can be proved only under the assumption that the viscosity is sufficiently high to counterbalance the external forces (cp. the section 6.3). This is standard from the viewpoint of the mathematical theory of the Navier-Stokes equations.

An important issue is of course the regularity. To try prove the existence of regular solutions to the time-dependent three-dimensional Navier-Stokes equations is not realistic in the scope of this PhD thesis. However, certain simplifications of the problem (P), that remain plausible models for the problem described in the section 2.2, lead to the existence of regular solutions (see the section 7.4). The issue is more easy in the case of (P_{st}) since classical regularity arguments are known for the stationary Navier-Stokes equations (see the section 6.4).

Finally, we would like to point out that the results of the chapters 4, 5 may interest the reader in their own right. The chapter is devoted to the study of the nonlocal radiation operators, and states new properties concerning coercivity, compactness in connection with nonsmooth interfaces and lower integrable heat sources. The chapter 5 gathers embedding results and coercivity inequalities involving the generalized operators curl and div, and states explicitely some consequences of new regularity results in this area.

Chapter 4

Auxiliary results I. The nonlocal radiation operators

The results of this section have been published in Druet [2009a].

Notations and assumptions: Throughout this section, we consider a bounded domain $\Omega \subset \mathbb{R}^3$ of the form described in the paragraph 3.1, that is,

$$\overline{\Omega} := \bigcup_{i=0}^{m} \overline{\Omega_i}\,,$$

where Ω_i are disjoint bounded domains that represent opaque bodies for $i = 1, \ldots, m$, and Ω_0 is an enclosed connected domain that represents a transparent medium. The assumption (3.1) is assumed to be satisfied with $\overline{\Omega_{\text{op}}} := \bigcup_{i=1}^{m} \overline{\Omega_i}$. We assume that all the domains involved are Lipschitzian. We denote by Σ the boundary $\partial\Omega_0$ of the cavity, and we assume that Σ belongs to $\mathcal{C}^1$ piecewise. We denote by S a surface measure on Σ. The unit normals on Σ, which we denote by $\vec{n}$, are defined almost everywhere in the sense of the measure S, and are piecewise continuous on Σ. In general, we shall consider the unit normal $\vec{n}$ that points *inward* to Ω_0.

Because of its importance we recall the definition (2.12) of the *view-factor*, i.e. the kernel of the operator K. For $(z,\, y) \in \Sigma \times \Sigma$, we set

$$w(z,y) := \begin{cases} \dfrac{\vec{n}(z)\cdot(y-z)\ \vec{n}(y)\cdot(z-y)}{\pi|y-z|^4}\,\Theta(z,y) & \text{if } z \neq y\,, \\ 0 & \text{if } z = y\,, \end{cases} \tag{4.1}$$

where Θ is the visibility function

$$\Theta(z,y) = \begin{cases} 1 & \text{if }]z,y[\, \subset \Omega_0\,, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

In these definitions, the symbol $]z,y[$ is an abbreviation for $\text{conv}\{z,y\}\setminus\{z,y\}$. We use the following notations. For Banach spaces X, Y, we denote by $\mathcal{L}(X,Y)$ the Banach space of the linear continuous mappings from X into Y. We denote by $\mathcal{K}(X,Y)$ the subspace of $\mathcal{L}(X,Y)$ that contains all mappings that are compact from X into Y. For $B \in \mathcal{L}(X,Y)$

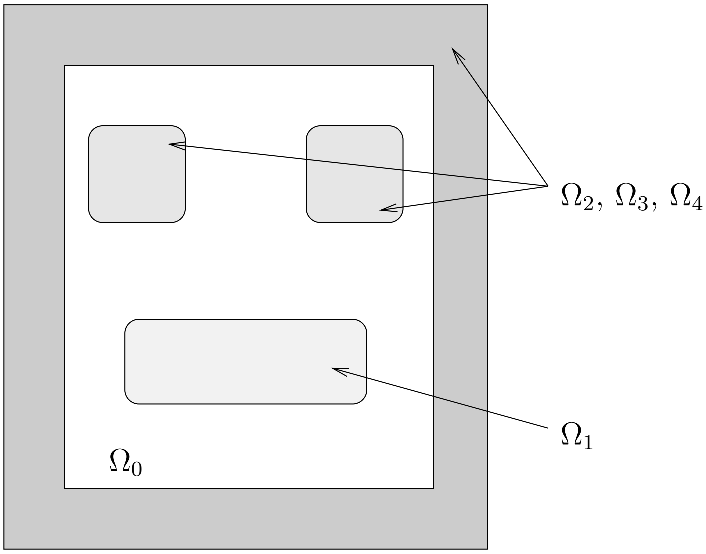


Figure 4.1: An enclosure Ω, with the transparent cavity Ω_0 and the opaque obstacles $\Omega_1, \ldots, \Omega_4$.

and $x \in X$, we denote by $B(x) \in Y$ the value of B at x. If B, $C \in \mathcal{L}(X, X)$, we denote by $B\,C(x)$ the element $B(C(x))$.

Outline of Chapter 4. We begin in section 4.1 by studying extensively the properties of the nonlocal operators

$$\begin{aligned} (K(f))(z) &:= \int_\Sigma w(z,y)\, f(y)\, dS_y \quad \text{for } z \in \Sigma\,, \\ G &:= (I - K)\,(I - (1-\epsilon)K)^{-1}\,\epsilon\,, \end{aligned} \tag{4.3}$$

which were introduced in (2.14) and (2.16) for the modeling of radiation. We then prove in the next section 4.2 some essential compactness properties of the operator K. Thanks to the properties established for K and G, new coercivity inequalities for the nonlinear forms occurring in the weak formulations (3.55) and (3.68) are derived in section 4.3.

Finally, we prove in section 4.4 two results about passage to the limit in an PDE with radiation boundary condition and right-hand side in L^1. This result is essential for the solution of the boundary value problems at the center of the thesis.

4.1 General properties of the operators K and G

We at first want to consider the integral operator K. A wide and profound knowledge about this integral operator K has been collected in the context of studies devoted to the *radiosity equation*, in particular on nonsmooth polyhedral surfaces (see Qatanani [1996], Rathsfeld [1999], Hansen [2002]).

Lemma 4.1.1. Assume that the general assumptions of Section 4 are satisfied. Then, for almost all $z \in \Sigma$, the set $\Sigma_z := \{y \in \Sigma \,:\, w(z,\, y) > 0\}$ of all points that can be seen from the point z is relatively open, and consequently a $\mathcal{C}^1$ piecewise surface itself

Proof. The surface Σ belongs to $\mathcal{C}^1$ piecewise, thus $\Sigma = \bigcup_{k=1}^{s} \overline{\Gamma_k}$ with $s > 0$ disjoint $\mathcal{C}^1$ surfaces $\Gamma_1, \ldots, \Gamma_k$. We consider an arbitrary point z located in the interior of one $\mathcal{C}^1$ piece of surface. We prove that $y \in \Sigma_z \cap \Gamma_k$, taken arbitrary, is an inner point of $\Sigma_z \cap \Gamma_k$, $k = 1, \ldots, s$.

Since $\vec{n}(z) \cdot (y - z)\ \vec{n}(y) \cdot (z - y)/\pi|y - z|^4$ is continuous on Γ_k, it remains positive in some ball $B_\rho(y) \cap \Sigma \subset \Gamma_k$. Therefore, we have only to show that the visibility function $\Theta(z, \cdot)$ remains positive in some ball centered at y.

Seeking a contradiction, suppose that $y_n \to y$, and that $\Theta(z, y_n) = 0$, that is, there exists some $\bar{y}_n \in \overline{\Omega_{\mathrm{op}}} \cap]z,\, y_n[$. Since $\Theta(z,\, y) = 1$, the sequence $\{\bar{y}_n\}$ cannot converge to a point in $]z,\, y[$, and therefore

$$\bar{y}_n \longrightarrow z \ \text{ (resp. } \bar{y}_n \longrightarrow y)$$

Denoting by Ω_z (resp. Ω_y) the part of Ω which contains z (resp. y), we note that the line $[z,\, y_n]$ crosses $\partial\Omega_z$ (resp. $\partial\Omega_y$) at the points $\{z_n,\, \bar{y}_n\}$ (resp. $\{\bar{y}_n,\, y\}$).

So $y_n - z$ (resp. $z - y_n$) converges to a vector tangential to $\partial\Omega_z$ at z (resp. to $\partial\Omega_y$ at y), and we must have

$$\vec{n}(z) \cdot (y - z)\ \vec{n}(y) \cdot (z - y) = 0\,.$$

Thus, y cannot belong to Σ_z, which is a contradiction.

The claim follows. □

The following Lemma has been proved in Hansen [2002] lemma 3.16 for a polyhedral boundary. Proofs in the case of a $\mathcal{C}^{1,\alpha}$ piecewise boundary have been given in Tiihonen [1997a] and Tiihonen [1997b]. We give a proof only for the convenience of the reader.

Lemma 4.1.2. Under the general assumptions of Section 4, the view-factor given by (4.1) satisfies the condition

$$\left\{\begin{array}{rcll} w(z,y) & = & w(y,z) & \text{for almost all } (z,y) \in \Sigma \times \Sigma\,, \\ w(z,y) & \geq & 0 & \text{for almost all } (z,y) \in \Sigma \times \Sigma\,, \\ \displaystyle\int_\Sigma w(z,y) dS_y & \leq & 1 & \text{for almost all } z \in \Sigma\,. \end{array}\right. \tag{4.4}$$

Proof. We use the method proposed in Tiihonen [1997b]. Since $\Sigma \in \mathcal{C}^1$ piecewise, the unit normals are almost everywhere defined on Σ. It follows that the kernel (4.1) is well defined, and obviously symmetric.

For $z \in \Sigma$, we introduce $\Sigma_z := \{y \in \Sigma \,:\, w(z,\, y) > 0\}$ which is nothing else but the set of all points that can be seen from the point z. Note that in view of Lemma 4.1.1, the set Σ_z consists of finitely many connected relatively open pieces of the surface Σ. We can write

$$\int_\Sigma w(z,y)\, dS_y = \int_{\Sigma_z} \frac{\vec{n}(z) \cdot (y - z)\, \vec{n}(y) \cdot (z - y)}{\pi|y - z|^4}\, dS_y\,.$$

We can equivalently express

$$\frac{\vec{n}(z) \cdot (y - z)\, \vec{n}(y) \cdot (z - y)}{\pi|y - z|^4} = -\frac{\cos(\phi_y)\ \cos(\phi_z)}{\pi|y - z|^2}\,, \tag{4.5}$$

where ϕ_y[resp. ϕ_z] is the angle between $\vec{n}(y)$[resp. $\vec{n}(z)$] and $(z-y)$[resp. $(y-z)$]. The representation (4.5) shows that w is invariant under rotations and translations. For this reason, the value of the integral remains unchanged if we assume that $z = 0$ and $\vec{n}(z) = (-1, 0, 0)$.

For obvious geometrical reasons, $\Sigma_z \subset \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,\big|\, x_1 < 0\}$. As a matter of fact, if $y \in \{x \in \mathbb{R}^3 \,\big|\, x_1 \geq 0\}$, then $]z, y[\not\subset \Omega_0$. For the same reason, we see that the line through the origin and an arbitrary point on the unit half-sphere $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 < 0\,,\, |x| = 1\}$ intersects Σ_z in at most one point. Passing to polar coordinates, we can parameterize the surface Σ_z with a mapping

$$\Psi : \mathcal{O} \subset \left]\frac{\pi}{2}, \pi\right[\times]0, 2\pi[\quad \longrightarrow \quad \Sigma_z\,,$$

$$y = \Psi(\phi_1, \phi_2) \quad := \quad \begin{pmatrix} r(\phi_1, \phi_2)\, \cos(\phi_1) \\ r(\phi_1, \phi_2)\, \sin(\phi_1)\, \cos(\phi_2) \\ r(\phi_1, \phi_2)\, \sin(\phi_1)\, \sin(\phi_2) \end{pmatrix},$$

where the parameterization domain $\mathcal{O}$ has to be adjusted to the surface. Note that the radius function r depends on the position of z relatively to Σ, but we can show that r is Lipschitz continuous. Denoting by the symbol G_Ψ the Gram determinant of the matrix Ψ', we can compute that

$$\vec{n}(\Psi) = \frac{1}{\left(r^2 \frac{\partial r}{\partial \phi_1}^2 \sin^2(\phi_1) + r^2 \frac{\partial r}{\partial \phi_2}^2 + r^4 \sin^2(\phi_1)\right)^{\frac{1}{2}}}$$

$$\times \begin{pmatrix} r^2\, \sin(\phi_1)\, \cos(\phi_1) + r\, \frac{\partial r}{\partial \phi_1}\, \sin^2(\phi_1) \\ r\, \frac{\partial r}{\partial \phi_2}\, \sin(\phi_2) - r\, \frac{\partial r}{\partial \phi_1} \sin(\phi_1)\, \cos(\phi_1)\, \cos(\phi_2) + r^2\, \sin^2(\phi_1)\, \cos(\phi_2) \\ -r\, \frac{\partial r}{\partial \phi_2}\, \cos(\phi_2) - r\, \frac{\partial r}{\partial \phi_1}\, \sin(\phi_1)\, \cos(\phi_1)\, \sin(\phi_2) + r^2\, \sin^2(\phi_1)\, \sin(\phi_2) \end{pmatrix},$$

$$\sqrt{G_\Psi} = \left(r^2 \frac{\partial r}{\partial \phi_1}^2 \sin^2(\phi_1) + r^2 \frac{\partial r}{\partial \phi_2}^2 + r^4 \sin^2(\phi_1)\right)^{\frac{1}{2}},$$

$$\vec{n}(\Psi) \cdot \Psi = \frac{r^3\, \sin(\phi_1)}{\left(r^2 \frac{\partial r}{\partial \phi_1}^2 \sin^2(\phi_1) + r^2 \frac{\partial r}{\partial \phi_2}^2 + r^4 \sin^2(\phi_1)\right)^{\frac{1}{2}}},$$

for λ_2-almost every $(\phi_1,\, \phi_2) \in]\frac{\pi}{2}, \pi[\times]0, 2\pi[$. We thus have that

$$w(z\,,\,\Psi) = \frac{(-n(\Psi) \cdot \Psi)\; (n(z) \cdot \Psi)}{\pi\, |\Psi|^4} = \frac{-r^4\, \sin(\phi_1)\, \cos(\phi_1)}{\pi\, r^4\, \left(r^2 \frac{\partial r}{\partial \phi_1}^2 \sin^2(\phi_1) + r^2 \frac{\partial r}{\partial \phi_2}^2 + r^4 \sin^2(\phi_1)\right)^{\frac{1}{2}}}.$$

Taking into consideration that $\phi_1 \in]\frac{\pi}{2}, \pi[$, this proves the nonnegativity of w.

We still have to compute the integral. If $\mathcal{O} =]\pi.\,\pi/2[\times]0,\,2\,\pi[$ we have

$$\begin{aligned}\int_\Sigma w(z,y)\,dS_y &= \int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi} w(z,\Psi)\,\sqrt{G_\Psi}\,d\phi_1\,d\phi_2\\ &= \int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi} \frac{-\sin(\phi_1)\,\cos(\phi_1)}{\pi}\,d\phi_1\,d\phi_2 = -\sin^2(\phi_1)\Big|_{\frac{\pi}{2}}^{\pi} = 1\,,\end{aligned}$$

proving (4.4). In the case of a smaller domain of parameterization $\mathcal{O}$, the integral is then less than or equal to one. □

The following property is to find in the literature, and can be verified easily:

Lemma 4.1.3. The domain Ω satisfies the enclosure property (3.1) if and only if for $S-$almost all $z \in \Sigma$ we have $\int_\Sigma w(z,y)\,dS_y = 1$.

Remark 4.1.4. If Ω is an enclosure, we can assume without loss of generality that the surface Σ *consists of one part*, i.e. that Σ is the boundary of a unique connected transparent cavity. Technically, we say if $A \subset \Sigma$ is such that for almost all $z \in A$, $\int_A w(z,y)\,dS_y = 1$, then we can assume that either $A = \Sigma$ or $A = \emptyset$.

In view of the integrability of w stated in Lemma 4.1.2, we see that the definition (2.14) of the operator K is well-posed at least for $f \in L^\infty(\Sigma)$. In the next Lemma, we recall the basic properties of the operator K that were proved in Tiihonen [1997a].

Lemma 4.1.5. (1) For every $1 \le p \le \infty$, the operator K extends to a linear bounded operator from $L^p(\Sigma)$ into itself.

(2) The norm estimate $\parallel K \parallel_{\mathcal{L}(L^p(\Sigma),L^p(\Sigma))} \le 1$ is valid.

(3) The operator K is positive, in the sense that $K(f) \ge 0$ almost everywhere on Σ if $f \ge 0$ almost everywhere on Σ; K is selfadjoint and positive semi-definite from $L^2(\Sigma)$ into itself.

(4) If the emissivity ϵ is a function such that (3.45) is satisfied, then for $1 \le p \le \infty$, the operator $(I - (1-\epsilon)K)$ has an inverse in $\mathcal{L}(L^p(\Sigma), L^p(\Sigma))$, with the representation

$$(I - (1-\epsilon)K)^{-1} = \sum_{i=0}^{\infty} (1-\epsilon)^i K^i\,. \tag{4.6}$$

Proof. See Tiihonen [1997a], Lemma 2 and 3. □

Actually, it is possible to prove a stronger statement than Lemma 4.1.5, (4).

Lemma 4.1.6. Assume that $\overline{\Omega} = \bigcup_{i=0}^m \overline{\Omega}_i$ with m disjoint polyhedra $\Omega_1, \ldots, \Omega_m$, and the cavity Ω_0. If the emissivity ϵ is not identically zero on Σ, then for $1 \le p \le \infty$, the operator $(I - (1-\epsilon)K)$ has an inverse in $\mathcal{L}(L^p(\Sigma), L^p(\Sigma))$.

Proof. This statement was proved in Laitinen and Tiihonen [2001], lemma 2. Since the compactness of K from $L^p(\Sigma)$ into itself is needed for the proof, its validity is restricted to $\mathcal{C}^{1,\alpha}$ surfaces (c.p. Lemma 4.2.2 below). We prove the lemma using a standard decomposition technique.

The invertibility of the operator $(I - (1-\epsilon)K)$ in $\mathcal{L}(L^p(\Sigma), L^p(\Sigma))$ is equivalent to the unique solvability in $L^p(\Sigma)$ of the radiosity equation

$$(I - (1-\epsilon)K)(f) = g\,, \tag{4.7}$$

for each given $g \in L^p(\Sigma)$. Observe that under the assumption (3.45), the unique solvability of (4.7) simply follows from the Neumann series theorem (c.p. Lemma 4.1.5, (4)), since

$$\|(1-\epsilon)\,K\|_{\mathcal{L}(L^p(\Sigma),L^p(\Sigma))} \le 1 - \epsilon_l < 1.$$

For $\rho > 0$ and $z \in \Sigma$, we introduce the notation $B_\rho(z) := \{y \in \Sigma\,:\, |z-y| \le \rho\}$. We at first prove the result assuming that there exists $\rho_0 > 0$ such that

$$\operatorname*{ess\,sup}_{z\in\Sigma} \int_{B_{\rho_0}(z)} w(z,\,y)\,dS_y < 1\,. \tag{4.8}$$

Assuming the validity of (4.8), we introduce the operators

$$(K_1(f))(z) := \int_{B_{\rho_0}(z)} w(z,\,y)\,f(y)\,dS_y\,, \quad (K_2(f))(z) := \int_{\Sigma\setminus B_{\rho_0}(z)} w(z,\,y)\,f(y)\,dS_y\,.$$

We verify easily by arguments similar to Lemma 4.1.5 that $K_1,\, K_2 \in \mathcal{L}(L^p(\Sigma), L^p(\Sigma))$. Under the assumption (4.8), we clearly obtain that $\|K_1\|_{\mathcal{L}(L^p(\Sigma),L^p(\Sigma))} < 1$. Thus, due to the Neumann-series theorem the operator $I - (1-\epsilon)\,K_1$ is invertible in $\mathcal{L}(L^p(\Sigma), L^p(\Sigma))$.

Applying the operator $(I - (1-\epsilon)\,K_1)^{-1}$ to the equation (4.7), we can equivalently express

$$(I - (I - (1-\epsilon)\,K_1)^{-1}\,(1-\epsilon)\,K_2)(f) = (I - (1-\epsilon)\,K_1)^{-1}(g)\,. \tag{4.9}$$

On the other hand, the integral operator K_2 has a kernel

$$k_2(z,\,y) = \chi_{\Sigma\setminus B_{\rho_0}(z)}(y)\,w(z,\,y)\,,$$

which is uniformely bounded by c/ρ_0^2. Thus, K_2 is a Fredholm operator and is compact from $L^p(\Sigma)$ into itself. The operator $(I - (1-\epsilon)\,K_1)^{-1}\,(1-\epsilon)\,K_2$ is as well compact, which proves the unique solvability of (4.9) and of (4.7).

It remains to prove that (4.8) is valid. Seeking a contradiction, assume that it is not the case. Then we can construct a sequence $\{z_n\} \subset \Sigma$ such that $\int_{B_{1/n}(z_n)} w(z_n,\,y)\,dS_y \ge 1 - 1/n$. In view of Lemma 4.4, it follows that

$$\int_{\Sigma\setminus B_{1/n}(z_n)} w(z_n,\,y)\,dS_y \le 1/n\,.$$

Fatou's lemma now implies that for almost all $y \in \Sigma$

$$\liminf_{n\to\infty} w(z_n,\,y) = \liminf_{n\to\infty} \chi_{\Sigma\setminus B_{1/n}(z_n)}(y)\,w(z_n,\,y) = 0\,.$$

On the other hand, since Σ is a compact set, there must exist $z^* \in \Sigma$, a vector $\vec{\xi}$ and a subsequence (not relabelled) such that $z_n \to z^*$ and $\vec{n}(z_n) \to \vec{\xi}$. For almost all $y \in \Sigma$ such that $\Theta(z^*, y) = 1$, we have

$$\vec{\xi} \cdot (y - z^*) \ \vec{n}(y) \cdot (z^* - y) = \liminf_{n \to \infty} \vec{n}(z_n) \cdot (y - z_n) \ \vec{n}(y) \cdot (z_n - y) = 0\,.$$

We now consider the set $A_1 := \{y \in \Sigma \,:\, \vec{\xi} \cdot (y - z^*)\}$. By definition, $A_1 \subset \{z^*\} + N_\xi$, where N_ξ is the plane $\{x \in \mathbb{R}^3 \,:\, \vec{\xi} \cdot x = 0\}$.

We consider an arbitrary face $\mathcal{P}$ of the polyhedron Ω. If $\mathcal{P} \subset \{z^*\} + N_\xi$, then $\Theta(z^*,\, y) = 0$ for all $y \in \mathcal{P}$, since two points located in the same plane face cannot see each other. Otherwise, the intersection $\mathcal{P} \cap \{z^*\} + N_\xi$ is either empty or a segment of line. Thus, the set A_1 consists at most of one face of the polyhedron that contains z^* and of $q - 1$ segments of line, where q denotes the total number of faces.

We now consider the set $A_2 := \{y \in \Sigma \,:\, \vec{n}(y) \cdot (z^* - y)\}$. We consider an arbitrary face $\mathcal{P}$ of the polyhedron. If $z^* \in \mathcal{P}$, then obviously $\mathcal{P} \subset A_2$, but $\Theta(z^*,\, y) = 0$ for all $y \in \mathcal{P}$. If $z^* \notin \mathcal{P}$, then $\mathcal{P} \cap A_2$ contains at most one point. Thus, the set A_2 consists of at most three faces of the polyhedron that contains z^* and of $q - 1$ isolated points.

We conclude that Σ has the representation

$$\Sigma = \{y \in \Sigma \,:\, \Theta(z^*,\, y) = 0\} \cup N\,,$$

with a set N of zero surface measure. Clearly, almost all points of Σ must belong to a common face with z^*, and Σ cannot be the boundary of a bounded domain Ω_0. □

Thanks to Lemma 4.1.5, we see that the operator G introduced in (2.16) is well-defined as an element of $\mathcal{L}(L^p(\Sigma), L^p(\Sigma))$. Note the following equivalent representations of the operator G:

$$G := (I - K)\,(I - (1 - \epsilon)K)^{-1}\epsilon = \epsilon - \epsilon K(I - (1 - \epsilon)K)^{-1}\epsilon \tag{4.10}$$

Lemma 4.1.7. (1) The operator $\mathbf{H} := I - G$ is positive and selfadjoint in $L^2(\Sigma)$. The operator G is itself selfadjoint.

(2) For $1 \le p \le \infty$, the norm estimate $\| \mathbf{H} \|_{\mathcal{L}(L^p(\Sigma), L^p(\Sigma))} \le 1$ is true.

Proof. See Laitinen and Tiihonen [1998]. □

In the next Lemma, we present some further elementary properties of G, K, and $\mathbf{H}$. They turn out to be essential for the discussion of the coercivity.

Lemma 4.1.8. (1) The equivalence $\mathbf{H}(\psi) = \psi \iff K(\psi) = \psi$ is valid.

(2) If $\psi \in L^p(\Sigma)$ $(1 < p \le \infty)$ satisfies $K(\psi) = \psi$, then ψ is a constant.

(3) If Ω is not an enclosure, then for $1 \le p \le \infty$, the strict estimate $\| \mathbf{H} \|_{\mathcal{L}(L^p(\Sigma), L^p(\Sigma))} < 1$ is true.

(4) If Ω is an enclosure, then $G(1)$ vanishes almost everywhere on Σ.

(5) Let Ω be an enclosure. For some $r+s \geq 1$ $(r,s>0)$, let $\psi \in L^{r+s}(\Sigma)$ satisfy $\int_\Sigma G(|\psi|^{r-1}\,\psi)\,|\psi|^{s-1}\,\psi = 0$. Then ψ is a constant.

(6) Let Ω be an enclosure. Define $\mathrm{sign}(0) := 0$. If $\psi \in L^1(\Sigma)$ satisfies $\int_\Sigma G(\psi)\,\mathrm{sign}(\psi) = 0$, then $\mathrm{sign}(\psi)$ is almost everywhere a constant on Σ.

Proof. (1): Assume first that $\mathbf{H}(\psi) = \psi$.

By definition, this means that $(1-\epsilon)\,\psi + \epsilon\,K\,(I-(1-\epsilon)K)^{-1}\,\epsilon\,(\psi) = \psi$, which implies that $K\,(I-(1-\epsilon)K)^{-1}\,\epsilon\,(\psi) = \psi$. Define

$$v \;:=\; (I-(1-\epsilon)K)^{-1}\epsilon\,(\psi)\,.$$

We then have $v-(1-\epsilon)K(v) = \epsilon\,\psi$ and $K(v) = \psi$. Hence $v = \psi$ and $K(\psi) = \psi$.

If we now start from $K(\psi) = \psi$, then we immediately see that $\epsilon K(\psi) = (I-(1-\epsilon)K)(\psi)$, so that $(I-(1-\epsilon)K)^{-1}\epsilon K(\psi) = \psi$. It follows that $\mathbf{H}(\psi) = (1-\epsilon)\,\psi + \epsilon K(\psi) = \psi$. This proves the first point.

(2): By assumption, we have for almost all $z \in \Sigma$ that $\psi(z) = \int_\Sigma w(z,y)\,\psi(y)\,dS_y$. First, let $p=2$. Then,

$$\begin{aligned} |\psi(z)|^2 = \left|\int_\Sigma w(z,y)\,\psi(y)\,dSy\right|^2 &\leq \left(\int_\Sigma w(z,y)\,dSy\right)\left(\int_\Sigma w(z,y)\,|\psi(y)|^2\,dSy\right) \\ &\leq \int_\Sigma w(z,y)\,|\psi(y)|^2\,dSy\,, \end{aligned} \tag{4.11}$$

by the triangle inequality, the Cauchy-Schwarz inequality, and the elementary properties 4.4 of the kernel w. Suppose now that there exists a set $M \subset \Sigma$ with positive surface measure such that strict inequality is valid. This would imply that

$$|\psi(z)|^2 < \int_\Sigma w(z,y)\,|\psi(y)|^2\,dSy \ \text{ on } M, \qquad |\psi(z)|^2 \leq \int_\Sigma w(z,y)\,|\psi(y)|^2\,dSy \ \text{ on } \Sigma\setminus M\,.$$

Integrating over Σ, it follows that

$$\begin{aligned} \int_\Sigma |\psi(z)|^2\,dSz &< \int_\Sigma \left(\int_M w(z,y)\,dSz + \int_{\Sigma\setminus M} w(z,y)\,dSz\right)|\psi(y)|^2\,dSy \\ &\leq \int_\Sigma |\psi(y)|^2\,dSy\,, \end{aligned}$$

which is a contradiction. Thus, for almost all $z \in \Sigma$ we must have the equality sign in (4.11).

This at first means that

$$\left|\int_\Sigma w(z,y)\,\psi(y)\,dSy\right| = \int_\Sigma w(z,y)\,|\psi(y)|\,dSy\,,$$

and for almost all $z \in \Sigma$ we must have

$$w(z,y)\,\psi(y)^- = 0, \qquad [\text{resp. } w(z,y)\,\psi(y)^+ = 0] \ \text{ for almost all } y \in \Sigma\,.$$

Without loss of generality, let $\psi^- = 0$.

Due to the equality in (4.11), we on the other hand have for almost all z the equality

$$\int_\Sigma w(z,y)^{1/2}\, w(z,y)^{1/2}\, \psi(y) dS_y = \left(\int_\Sigma w(z,y) dS_y\right)^{\frac{1}{2}} \left(\int_\Sigma w(z,y)\, \psi^2(y) dS_y\right)^{\frac{1}{2}} .$$

By a well-known property of the Cauchy-Schwarz inequality, this implies that

$$w(z,y)^{\frac{1}{2}} = \lambda(z)\, w(z,y)^{\frac{1}{2}}\, \psi(y) ,$$

with a real number $\lambda(z)$, for almost all z. Thus, for almost all y and z that can see each other, we get $\psi(y) = \lambda(z)^{-1}$, which obviously leads to the claim.

In the case $1 < p < 2$, we can argue just the same. For almost all $z \in \Sigma$, we must have the equation

$$\int_\Sigma w(z,y)^{\frac{1}{p'}}\, w(z,y)^{\frac{1}{p}}\, \psi(y)\, dSy = \left(\int_\Sigma w(z,y)\, dSy\right)^{\frac{p}{p'}} \left(\int_\Sigma w(z,y)\, |\psi(y)|^p\, dSy\right) ,$$

which implies, with some $\lambda(z)$, the equality $w(z,y)^{\frac{1}{p'}} = \lambda(z) \left[w(z,y)^{\frac{1}{p}}\, \psi(y)\right]^{\frac{p}{p'}}$. The claim follows.

(3): The third claim was proved in Tiihonen [1997b], Laitinen and Tiihonen [1998]. We give an analogous simpler proof. Since Ω is no enclosure, we have $K(1) \not\equiv 1$. Thus, by (2), there exists no $\psi \in L^2(\Sigma)$ such that $K(\psi) = \psi$. By (1), we obtain that also $\mathbf{H}(\psi) \neq \psi$ for all $\psi \in L^2(\Sigma)$, i. e. 1 is not an eigenvalue of $\mathbf{H}$. But as $\mathbf{H}$ is selfadjoint in $L^2(\Sigma)$, $\| \mathbf{H} \|_{\mathcal{L}(L^2(\Sigma),L^2(\Sigma))}$ must be an eigenvalue of $\mathbf{H}$. It follows that

$$\| \mathbf{H} \|_{\mathcal{L}(L^2(\Sigma),L^2(\Sigma))} < 1 ,$$

and by classical interpolation arguments for linear positive operators, the claim even follows for all $1 \le p \le \infty$.

(4): If Ω is an enclosure, then $K(1) = 1$ almost everywhere on Σ in view of Lemma 4.4. Thus, by point (1), $\mathbf{H}(1) = 1$. The claim follows.

(5): By the triangle inequality and Hölder's inequality, we at first have

$$\begin{aligned} 0 &= \int_\Sigma G(|\psi|^{r-1}\,\psi)\,|\psi|^{s-1}\,\psi \ge \int_\Sigma |\psi|^{r+s} - \left|\int_\Sigma \mathbf{H}(|\psi|^{r-1}\,\psi)\,|\psi|^{s-1}\,\psi\right| \\ &\ge \int_\Sigma |\psi|^{r+s} - \int_\Sigma \left|\mathbf{H}(|\psi|^{r-1}\,\psi)\right|\,|\psi|^s \\ &\ge \int_\Sigma |\psi|^{r+s} - \| \mathbf{H}(|\psi|^{r-1}\,\psi) \|_{L^{\frac{r+s}{r}}(\Sigma)} \| \, |\psi|^s \, \|_{L^{\frac{r+s}{s}}(\Sigma)} \\ &\ge (1 - \| \mathbf{H} \|_{\mathcal{L}(L^{\frac{r+s}{r}}(\Sigma),L^{\frac{r+s}{r}}(\Sigma))}) \int_\Sigma |\psi|^{r+s} . \end{aligned} \tag{4.12}$$

Thus, we must have everywhere the equality sign. This at first means that

$$\mathbf{H}(|\psi|^{r-1}\,\psi)\,|\psi|^{s-1}\,\psi \ge 0, \qquad [\text{resp. } \mathbf{H}(|\psi|^{r-1}\,\psi)\,|\psi|^{s-1}\,\psi \le 0] \;\text{ a. e. on } \Sigma , \tag{4.13}$$

and, at second, that we have in particular

$$\int_\Sigma \mathbf{H}(|\psi|^{r-1}\,\psi)\,|\psi|^s = \|\,\mathbf{H}(|\psi|^{r-1}\,\psi)\,\|_{L^{\frac{r+s}{r}}(\Sigma)}\,\|\,|\psi|^s\,\|_{L^{\frac{r+s}{s}}(\Sigma)}\;.$$

The latter point immediately implies that

$$|\mathbf{H}(|\psi|^{r-1}\,\psi)| = c\,[|\psi|^s]^{\frac{(r+s)/s}{(r+s)/r}} = c\,|\psi|^r\,. \tag{4.14}$$

Because of (4.12), we clearly have $|c| \leq 1$. Since $-1 \leq c < 1$ implies $\psi \equiv 0$ again by (4.12), we just have to discuss the case $c = 1$.

Now, (4.14) gives that $|\psi|^r = |\mathbf{H}(|\psi|^{r-1}\,\psi)| \leq \mathbf{H}(|\psi|^r)$, so by definition $G(|\psi|^r) \leq 0$. Since Ω is an enclosure, $G(1) = 0$. By the fact that G is selfadjoint, we can write $0 \geq \int_\Sigma G(|\psi|^r) = \int_\Sigma G(1)\,|\psi|^r = 0$. The first and second points of this lemma now imply that $|\psi|^r \equiv C^r$, for some positive constant C.

Returning to (4.14), where we can assume $c = 1$, with this information, we get that $|\psi| = C = |\mathbf{H}(\psi)|$. Using (4.13), we have in addition $\text{sign}(\mathbf{H}(\psi)) = \pm\text{sign}(\psi)$. Thus, $\mathbf{H}(\psi) = \pm\psi$.

Again, because of the first line in relation (4.12), we see that $\mathbf{H}(\psi) = -\psi$ implies that $\psi = 0$. On the other hand, because of (1) and (2), $\mathbf{H}(\psi) = \psi$ implies that ψ is constant. This proves point (5).

(6): Observe that

$$\psi\,G(\text{sign}(\psi)) = |\psi| - \psi\,H(\text{sign}(\psi)) \geq (1 - \|H\|_{\mathcal{L}(L^\infty(\Sigma),L^\infty(\Sigma))})\,|\psi| \geq 0\,,$$

almost everywhere on Σ. On the other hand, since G is selfadjoint, we have

$$0 = \int_\Sigma G(\psi)\,\text{sign}(\psi) = \int_\Sigma \psi\,G(\text{sign}(\psi)) \geq 0\,,$$

and we see that $\psi\,G(\text{sign}(\psi))$ vanishes almost everywhere on Σ. This means that $|\psi| = \psi\,H(\text{sign}(\psi))$, and we deduce that

$$H(\text{sign}(\psi)) = \text{sign}(\psi) \quad \text{for almost all } z \in \Sigma \text{ such that } |\psi(z)| > 0\,.$$

In particular, we have for $z \in \Sigma$ such that $\psi(z) > 0$

$$1 = H(\text{sign}(\psi))(z) = H(\chi_{\{z\in\Sigma\,:\,\psi>0\}})(z) - H(\chi_{\{z\in\Sigma\,:\,\psi<0\}})(z)\,.$$

Since H is a positive operator, the last idendity is only possible assuming that for almost all $z \in \Sigma$ such that $\psi(z) > 0$

$$1 = H(\chi_{\{z\in\Sigma\,:\,\psi>0\}})(z), \quad 0 = H(\chi_{\{z\in\Sigma\,:\,\psi<0\}})(z)\,.$$

Thus, we can write that

$$H(\chi_{\{z\in\Sigma\,:\,\psi>0\}}) \geq \chi_{\{z\in\Sigma\,:\,\psi>0\}} \quad \text{almost everywhere on } \Sigma\,,$$

and it follows that $G(\chi_{\{z\in\Sigma\,:\,\psi>0\}}) \leq 0$ on Σ. But $G(\chi_{\{z\in\Sigma\,:\,\psi>0\}})$ has mean-value zero on Σ, and thus, $G(\chi_{\{z\in\Sigma\,:\,\psi>0\}}) = 0$ almost everywhere on Σ. Owing to (1) and (2), it follows that $\chi_{\{z\in\Sigma\,:\,\psi>0\}}$ is almost everywhere a constant. Analogously, we can deduce that $\chi_{\{z\in\Sigma\,:\,\psi<0\}}$ is almost everywhere a constant. The claim follows. $\square$

We recall that for Banach spaces X, Y, we denote by $\mathcal{K}(X, Y)$ the set of all linear bounded compact mappings from X into Y.

Lemma 4.1.9. (1) Let $\Sigma \in \mathcal{C}^{1,\alpha}$. For $1 < p < \infty$, the operator K belongs to the class $\mathcal{K}(L^p(\Sigma), L^p(\Sigma))$.

(2) Let $\Sigma \in \mathcal{C}^{1,\alpha}$ have at least one edge. Then, for $1 < p \leq \infty$, we have $K \notin \mathcal{K}(L^p(\Sigma), L^p(\Sigma))$.

Proof. (1): This assertion was stated in Tiihonen [1997a], Laitinen and Tiihonen [1998] and follows from classical arguments about weakly singular integral operators. For a more detailed proof, see the next section. (2): This fact is also well-known. The reader will find an elementary counter-example in the next section. □

For the discussion of L^1 right-hand sides, another compactness property of K turns out to be important.

Lemma 4.1.10. Let $\Sigma \in \mathcal{C}^{1,\alpha}$. Then for $\frac{1}{\alpha} < p$, we have $K \in \mathcal{K}(L^p(\Sigma), C(\Sigma))$.

Proof. In the case that Σ is the boundary of a convex domain Ω_0, the continuity and the compactness of K into $C(\Sigma)$ follow from standard arguments about weakly singular integral operators (see for example the part about Schur integral operators of the book Alt [1985]). The proof relies on the one hand on the estimate

$$|(K(f))(z_1) - (K(f))(z_2)| \leq \|f\|_{L^p(\Sigma)} \left(\int_\Sigma |w(z_1, y) - w(z_2, y)|^{p'} \, dS_y \right)^{1/p'},$$

for $z_1,\ z_2 \in \Sigma$, and on the other hand on the uniform continuity

$$\max_{|z_1 - z_2| \leq \delta} \left(\int_\Sigma |w(z_1, y) - w(z_2, y)|^{p'} \, dS_y \right)^{1/p'} \xrightarrow{\delta \to 0} 0\,.$$

Due to the discontinuous visibility function Θ in the definition (4.1) of the kernel w, the proof is slightly more involved in the case of nonconvex Σ.

We introduce the notation $\tilde{w}(z, y) := \vec{n}(z) \cdot (y - z)\, \vec{n}(y)\, (z - y)/\pi\, |z - y|^4$. We then have, by the triangle inequality,

$$\int_\Sigma |w(z_1, y) - w(z_2, y)|^{p'} \, dS_y \tag{4.15}$$

$$\leq \int_\Sigma |\tilde{w}(z_1, y) - \tilde{w}(z_2, y)|^{p'}\, \Theta(z_1, y)\, dS_y + \int_\Sigma |\Theta(z_1, y) - \Theta(z_2, y)|^{p'}\, |\tilde{w}(z_2, y)|^{p'} \, dS_y\,. \tag{4.16}$$

Since $\tilde{w}$ is a weakly singular kernel, the standard arguments give

$$\max_{|z_1 - z_2| \leq \delta} \int_\Sigma |\tilde{w}(z_1, y) - \tilde{w}(z_2, y)|^{p'} \, dS_y \xrightarrow{\delta \to 0} 0\,.$$

On the other hand, we have

$$\int_\Sigma |\Theta(z_1,\, y) - \Theta(z_2,\, y)|^{p'}\, |\tilde{w}(z_2,\, y)|^{p'}\, dS_y \;= \int_{A(z_1,\, z_2)} |\tilde{w}(z_2,\, y)|^{p'}\, dS_y\,,$$

with $A(z_1,\, z_2) := \{y \in \Sigma \,:\, \Theta(z_1,\, y) \neq \Theta(z_2,\, y)\}$. We can further estimate

$$\int_{A(z_1,\, z_2)} |\tilde{w}(z_2,\, y)|^{p'}\, dS_y \;\leq \left(\int_\Sigma |\tilde{w}(z_2,\, y)|^{p'\, q}\, dS_y\right)^{1/q} \operatorname{meas}(A(z_1,\, z_2))^{1/q'}\,.$$

Choosing $q\, p'\, \alpha < 1$, and a $\gamma > 0$, we can write that

$$\begin{aligned}
&\int_\Sigma |\tilde{w}(z_2,\, y)|^{p'\, q}\, dS_y = \int_{B_\gamma(z_2)} |\tilde{w}(z_2,\, y)|^{p'\, q}\, dS_y + \int_{\Sigma\setminus B_\gamma(z_2)} |\tilde{w}(z_2,\, y)|^{p'\, q}\, dS_y \\
&\leq C\, \gamma^{2-2\, q\, p'\, \alpha} + \operatorname{meas}(\Sigma \setminus B_\gamma(z_2))\, \frac{C}{\gamma^{2\, p'\, q}} \\
&\leq C\,,
\end{aligned}$$

with a constant independent of γ. Therefore

$$\int_\Sigma |\Theta(z_1,\, y) - \Theta(z_2,\, y)|^{p'}\, |\tilde{w}(z_2,\, y)|^{p'}\, dS_y \leq C\, \operatorname{meas}(A(z_1,\, z_2))^{1/q'}\,,$$

and it remains to show that

$$\max_{|z_1 - z_2| \leq \delta} \operatorname{meas}(A(z_1,\, z_2)) \xrightarrow{\delta \to 0} 0\,.$$

This is the object of the following Lemma 4.1.11. □

Lemma 4.1.11. Assume that $\Sigma \in \mathcal{C}^{1,\alpha}$. For $z_1,\, z_2 \in \Sigma$, define

$$A(z_1,\, z_2) := \{y \in \Sigma \,:\, \Theta(z_1,\, y) \neq \Theta(z_2,\, y)\}\,.$$

Then, we have

$$\max_{z_1,\, z_2 \in \Sigma\, :\, |z_1 - z_2| \leq \delta} \operatorname{meas}(A(z_1,\, z_2)) \leq c\, \delta^\alpha\,.$$

Proof. Since the proof is quite technical, we give it in the section 4.2 below. □

Remark 4.1.12. The simple two dimensional example of Figure 4.2 below shows that the property stated in Lemma 4.1.11 does not hold true on polyhedral domains. We always can find $z_1,\, z_2 \in \partial\Omega_2$ such that $|z_1 - z_2| \leq \delta$ and $\operatorname{meas}(A(z_1,\, z_2)) \geq \operatorname{meas}(\partial\Omega_1) - \gamma_0$.

Remark 4.1.13. The remark 4.1.12 shows that the properties of the operator K on spaces of continuous functions for nonsmooth surfaces is a delicate topic. The chapter 3 of Hansen [2002] is devoted to this question in the case of polyhedral surfaces.

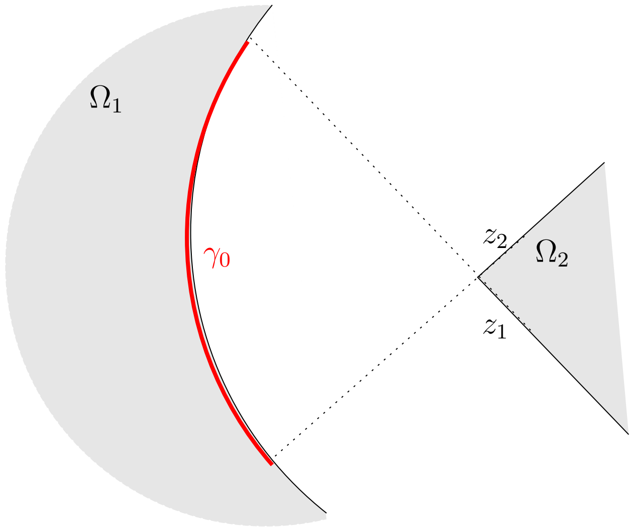


Figure 4.2: meas$(A(z_1, z_2)$ remains constant as $|z_1 - z_2| \to 0$ along the edges.

If Σ is the boundary of a convex polyhedron, then one can show that

$$K \in \mathcal{L}(C_p(\Sigma), C_p(\Sigma)) , \tag{4.17}$$

where $C_p(\Sigma)$ is the space of functions that are uniformely continuous on each face of Ω. Due to the visibility function Θ, the property (4.17) fails in general on nonconvex polyhedra. We only have

$$K \in \mathcal{L}(C_r(\Sigma), C_r(\Sigma)) , \tag{4.18}$$

where $C_r(\Sigma)$ denotes the space of functions that are continuous in the interior of each face, but not in the vertices of the polyhedron.

Lemma 4.1.14. (1) Introduce an operator $\tilde{\mathbf{H}}$ by $G = \epsilon\,(I - \tilde{\mathbf{H}})$. If $\Sigma \in \mathcal{C}^{1,\alpha}$, then for $1 < p < \infty$, the operator $\tilde{\mathbf{H}}$ belongs to $\mathcal{K}(L^p(\Sigma), L^p(\Sigma))$.

(2) For $1 \leq p \leq \infty$ let $1/p + 1/p' = 1$. Define $\tilde{\mathbf{H}}_p := \epsilon^{\frac{1}{p}}\, K\,(I - (1-\epsilon)K)^{-1}\,\epsilon^{\frac{1}{p'}}$. Then, the norm estimate $\|\tilde{\mathbf{H}}_p\|_{\mathcal{L}(L^p(\Sigma), L^p(\Sigma))} \leq 1$ is valid.

(3) Let $\psi \in L^\infty(\Sigma)$ satisfy $|\psi| \leq 1$ almost everywhere on Σ. Then, if neither $\psi = 1$ nor $\psi = -1$ almost everywhere on Σ, we must have $|\tilde{\mathbf{H}}(\psi)| < 1$ almost everywhere on Σ.

(4) If $\Sigma \in \mathcal{C}^{1,\alpha}$, then the operator $\tilde{\mathbf{H}}$ is weakly sequentially compact from $L^1(\Sigma)$ into itself.

Proof. (1): The first claim follows from representation (4.10) and Lemma 4.1.9, since $\tilde{\mathbf{H}} = K\,(I - (1-\epsilon)K)^{-1}\,\epsilon$.

(2): We readily verify that

$$(I - K(1-\epsilon))^{-1}\,K \quad = \quad K\,(I - (1-\epsilon)K)^{-1}\,. \tag{4.19}$$

For an arbitrary $f \in L^2(\Sigma)$, define $(I-(1-\epsilon)K)^{-1}(f) =: v$. Then, the equality $(1-\epsilon)K(v) = v - f$ is obviously true. This enables us to write that

$$\Big[\big(I - K(1-\epsilon)\big)\, K\Big](v) = K(v) - K\big((1-\epsilon)K(v)\big) = K(v) - (K(v) - K(f)) \\ = K(f)\,.$$

It follows that $(I - K(1-\epsilon))\, K\, (I-(1-\epsilon)K)^{-1}\,(f) = K(f)$, which proves (4.19).

We at first consider the case $1 < p < \infty$.

By definition, we have $\tilde{\mathbf{H}}_p = \epsilon^{\frac{1}{p}} K\,(I-(1-\epsilon)K)^{-1}\,\epsilon^{\frac{1}{p'}}$, and because of the relation (4.19), we can also write this in the form

$$\tilde{\mathbf{H}}_p = \epsilon^{\frac{1}{p}} (I - K(1-\epsilon))^{-1}\, K\, \epsilon^{\frac{1}{p'}}\,.$$

For an arbitrary $f \in L^p(\Sigma)$, we define

$$R := \big[\epsilon^{\frac{1}{p}} (I - K(1-\epsilon))^{-1}\, K\, \epsilon^{\frac{1}{p'}}\big](f)\,.$$

This definition allows to write that $\frac{R}{\epsilon^{1/p}} - K\left(\frac{1-\epsilon}{\epsilon^{1/p}}\, R\right) = K\big(\epsilon^{1/p'} f\big)$, which is equivalent to the equality $\frac{R}{\epsilon^{1/p}} = K\Big(\epsilon^{1/p'}\, f + \frac{1-\epsilon}{\epsilon^{1/p}}\, R\Big)$. Thus, using the fact that $\|K\|_{\mathcal{L}(L^p(\Sigma),\, L^p(\Sigma))} \le 1$, we deduce the inequality

$$\int_\Sigma \frac{|R|^p}{\epsilon} = \int_\Sigma \left| K\Big(\epsilon^{1/p'}\, f + \frac{1-\epsilon}{\epsilon^{1/p}}\, R\Big)\right|^p \le \int_\Sigma \left|\epsilon^{\frac{1}{p'}}\, f + \frac{(1-\epsilon)}{\epsilon^{1/p}} R\right|^p \\ = \int_\Sigma \frac{1}{\epsilon}\; |\epsilon\, f + (1-\epsilon)\, R|^p\,.$$

Using the convexity of the function $g(s) = s^p$ and the triangle inequality, we obtain that

$$\int_\Sigma \frac{|R|^p}{\epsilon} \le \int_\Sigma \frac{1}{\epsilon}\,(\epsilon\,|f|^p + (1-\epsilon)\,|R|^p)\,.$$

It follows that

$$\|\, \tilde{\mathbf{H}}_p(f)\,\|^p_{L^p(\Sigma)} = \|\, R\,\|^p_{L^p(\Sigma)} \le \|\, f\,\|^p_{L^p(\Sigma)}\,,$$

proving the result. The cases $p = 1$ and $p = \infty$ are straightforward exercises.

(3): Consider an arbitrary function $\psi \in L^\infty(\Sigma)$ such that $|\psi| \le 1$ almost everywhere on Σ. We introduce two functions R, J by

$$R = \epsilon\,\psi + (1-\epsilon)J, \qquad J = K(R)\,. \tag{4.20}$$

Note that $\tilde{\mathbf{H}}(\psi) = J$. In view of (2), we thus have $|J| \le 1$ almost everywhere on Σ. Our definition (4.20) obviously implies the set identity

$$A := \Big\{z \in \Sigma\,:\, R(z) = 1\Big\} = \Big\{z \in \Sigma\,:\, R(z) = 1 = \psi(z) = J(z)\Big\}\,. \tag{4.21}$$

Taking $z \in A$ arbitrary, we can write, on the other hand,

$$1 = J(z) = \int_{\Sigma} w(z,y)\, R(y)\, dS_y = \int_A w(z,y)\, dS_y + \int_{\{R<1\}} w(z,y)\, R(y)\, dS_y\,. \tag{4.22}$$

The latter equality is only possible if $\int_A w(z,y)\, dS_y = 1$. Since this is valid for any $z \in A$, we have by definition that the set A sees only itself. Therefore, by Remark 4.1.4, it follows either that $\mathrm{meas}(A) = 0$, or that $\mathrm{meas}(\Sigma \setminus A) = 0$.

Assume finally that $J(z) = 1$ for a $z \in \Sigma$. Writing (4.22) in this point gives a contradiction if $\mathrm{meas}(A) = 0$. This means that either $\tilde{\mathbf{H}}(\psi)(z) = J(z) < 1$ a. e. on Σ or $\mathrm{meas}(\Sigma \setminus A) = 0$.

We can argue analogously with the set $B := \{z \in \Sigma \,:\, R(z) = -1\}$. We conclude that if neither A nor B are the whole of Σ, then they must both have zero measure, and that $-1 < \tilde{\mathbf{H}}(\psi) < 1$ a. e. on Σ, proving the claim.

(4): This is an easy consequence of Lemma 4.1.10. Consider a sequence $\{\psi_n\} \subset L^1(\Sigma)$ such that $\|\psi_n\|_{L^1(\Sigma)} \le C$, and choose an arbitrary measurable subset $A \subset \Sigma$. Then

$$\begin{aligned}\int_A |\tilde{\mathbf{H}}(\psi_n)| \le \int_A \tilde{\mathbf{H}}(|\psi_n|) &= \int_{\Sigma} ((I - (1-\epsilon)K)^{-1}\, \epsilon)(|\psi_n|)\, K(\chi_A) \\ &\le c\, \|K(\chi_A)\|_{C(\Sigma)}\, \|\psi_n\|_{L^1(\Sigma)}\,.\end{aligned}$$

It follows that as $\mathrm{meas}(A) \to 0$

$$\sup_{n\in\mathbb{N}} \int_A |\tilde{\mathbf{H}}(\psi_n)| \to 0,$$

proving the equi-integrability of the sequence $\{\tilde{\mathbf{H}}(\psi_n)\}$. This proves the weak compactness in L^1. $\square$

4.2 Compactness of the operator K

$\mathcal{C}^{1,\alpha}$-boundary For the following definition, see for example Alt [1985].

For a bounded domain $\Omega \subset \mathbb{R}^n$, we write $\partial\Omega \in \mathcal{C}^{1,\alpha}$ if and only if for a $m \in \mathbb{N}$ and for $j = 1, \ldots, m$ there exists numbers r_j, $h_j > 0$, a function $g_j \in C^{1,\alpha}(\mathbb{R}^{n-1})$, and an Euclidean coordinate system $e_1^j, \ldots, e_n^j$ in $\mathbb{R}^n$ such that the following holds.

For $x \in \mathbb{R}^n \,:\, x = \sum_{i-1}^n x_i^j\, e_i^j$ and $(x_1^j, \ldots, x_{n-1}^j) =: x'^j$, if $|x'^j| < r_j$, we have

$$\begin{aligned} x_n^j = g_j(x'^j) &\Rightarrow x \in \partial\Omega\,, \\ 0 < x_n^j - g_j(x'^j) < h_j &\Rightarrow x \in \Omega\,, \\ -h_j < x_n^j - g_j(x'^j) < 0 &\Rightarrow x \notin \Omega\,. \end{aligned}$$

The sets $U_j := \{x \in \mathbb{R}^n \,:\, |x'^j| < r_j,\, |x_n^j - g_j(x'^j)| < h_j\}$ are an open covering of $\partial\Omega$. We can complete it to an open covering $U_0, \ldots, U_m$ of $\overline{\Omega}$ by choosing a suitable open set $U_0 \subset\subset \Omega$. We can also choose a partition of the unity $\eta_0, \ldots, \eta_m$ subordinated to this covering.

Setting

$$\Psi_j(t) := \sum_{i=1}^{n-1} t_i\, e_i^j + g_j(t)\, e_n^j \text{ for } t \in \mathbb{R}^{n-1}\,,$$

and

$$\mathcal{O}_j := \{t \in \mathbb{R}^{n-1} \,:\, |t| < r_j\}\,,$$

we have, for $f \in L^1(\partial\Omega)$,

$$\int_{\partial\Omega} f dS \;:=\; \sum_{j=1}^{m} \int_{\mathbb{R}^{n-1}} (\eta_j\, f)(\Psi_j(t))\,\sqrt{1+|\nabla g_j(t)|^2}\,dt\,.$$

We write $\partial\Omega \in \mathcal{C}^{1,\alpha}$ *piecewise* if and only if $\partial\Omega = \bigcup_{i=1}^{k} \Gamma_s$ with relatively open sets Γ_i being $\mathcal{C}^{1,\alpha}-$surfaces themselves.

Through this section we will assume that the local Euclidean coordinate systems $e_1^j, \ldots, e_n^j$ can be transformed into the standard basis by means of an orthogonal mapping.

Though the following result was stated in Tiihonen [1997a], Tiihonen [1997b], Laitinen and Tiihonen [1998], Laitinen and Tiihonen [2001], we prove it again in somewhat greater detail.

Lemma 4.2.1. Let $\Sigma \in \mathcal{C}^{1,\alpha}$. Then, for $1 < p < \infty$,

$$K \in \mathcal{K}(L^p(\Sigma), L^p(\Sigma))\,.$$

Proof. In the following, $n = 3$.

First step For $(t,s) \in \mathbb{R}^{n-1}$, and for $i,j = 1,\ldots,m$ define

$$k_{i,j}(t,s) := \begin{cases} \eta_i^{1/p'}(\Psi_i(s))\;\eta_j^{1/p}(\Psi_j(t))\;w(\Psi_j(t),\Psi_i(s)) & \text{if } (t,s) \in \mathcal{O}_j \times \mathcal{O}_i\,, \\ 0 & \text{else }\,, \end{cases}$$

and for $f \in L^\infty(\mathbb{R}^{n-1})$, define an operator $\tilde{K}_{i,j}$ by

$$\Big(\tilde{K}_{i,j}(f)\Big)(t) := \int_{\mathbb{R}^{n-1}} k_{i,j}(t,s)\, f(s)\, ds\,.$$

We have

$$\begin{aligned} &\left|\Big(\tilde{K}_{i,j}(f)\Big)(t)\right|^p \\ &\quad= \left|\int_{\mathcal{O}_i} \big[\eta_i(\Psi_i(s))\;w(\Psi_j(t),\Psi_i(s))\big]^{1/p'} \big[\eta_j(\Psi_j(t))\;w(\Psi_j(t),\Psi_i(s))\big]^{1/p} f(s)\,ds\right|^p \\ &\quad\le \left(\int_{\mathcal{O}_i} \eta_i(\Psi_i(s))\,w(\Psi_j(t),\Psi_i(s))\,ds\right)^{p/p'} \left(\int_{\mathcal{O}_i} \eta_j(\Psi_j(t))\,w(\Psi_j(t),\Psi_i(s))\,|f(s)|^p\,ds\right)\,. \end{aligned}$$

We can estimate

$$\int_{\mathcal{O}_i} \eta_i(\Psi_i(s))\, w(\Psi_j(t), \Psi_i(s))\, ds \le \int_{\mathcal{O}_i} \eta_i(\Psi_i(s))\, w(\Psi_j(t), \Psi_i(s))\, \sqrt{1 + |\nabla g_i(s)|^2} ds$$
$$\le \int_{\Sigma} w(\Psi_j(t), z)\, dS_z \le 1\,.$$

Thus,

$$\int_{\mathbb{R}^{n-1}} \Big| \big(\tilde{K}_{i,j}(f)\big)(t) \Big|^p dt \le \int_{\mathbb{R}^{n-1}} \left(\int_{\mathcal{O}_i} \eta_j(\Psi_j(t))\; w(\Psi_j(t), \Psi_i(s))\, |f(s)|^p\, ds \right) dt$$
$$= \int_{\mathcal{O}_i} |f(s)|^p \left(\int_{\mathcal{O}_j} \eta_j(\Psi_j(t))\; w(\Psi_j(t), \Psi_i(s))\, dt \right) ds \le \| f \|^p_{L^p(\mathbb{R}^{n-1})}\,.$$

By this inequality, $\tilde{K}_{i,j}$ extends to an element of $\mathcal{L}(L^p(\mathbb{R}^{n-1}), L^p(\mathbb{R}^{n-1}))$.

Second step: estimates on the kernel It holds that

$$|k_{i,j}(t,s)| \;=\; |\eta_i^{1/p'}(\Psi_i(s))\; \eta_j^{1/p}(\Psi_j(t))\; w(\Psi_j(t), \Psi_i(s))| \;\le\; w(\Psi_j(t), \Psi_i(s))\,,$$

in $\mathcal{O}_j \times \mathcal{O}_i$.

Let $\mu_0 > 0$ be a number that we fix later.

In the sets $\{(t,s) \in \mathcal{O}_j \times \mathcal{O}_i \;:\; |\Psi_j(t) - \Psi_i(s)| \ge \mu_0\}$, we have the estimate

$$\begin{aligned} w(\Psi_j(t), \Psi_i(s)) \;&=\; \frac{\vec{n}(\Psi_j(t)) \cdot (\Psi_i(s) - \Psi_j(t))\; \vec{n}(\Psi_i(s)) \cdot (\Psi_j(t) - \Psi_i(s))}{\pi\, |\Psi_j(t) - \Psi_i(s)|^{n+1}} \\ &\le\; \frac{\mathrm{diam}^2(\Omega)}{\pi \mu_0^{n+1}}\,. \end{aligned}$$

In the sets $\{(t,s) \in \mathcal{O}_j \times \mathcal{O}_i \;:\; |\Psi_j(t) - \Psi_i(s)| < \mu_0\}$, we have, first supposing $i = j$,

$$w(\Psi_i(t), \Psi_i(s)) \;=\; \frac{\vec{n}(\Psi_i(t) \cdot (\Psi_i(s) - \Psi_i(t)\; \vec{n}(\Psi_i(s)) \cdot (\Psi_i(t) - \Psi_i(s))}{\pi\, |\Psi_i(t) - \Psi_i(s)|^{n+1}}\,.$$

This expression is invariant under translations and rotations. By assumption, the local coordinate system can be transformed by an orthogonal mapping into the standard basis in $\mathbb{R}^n$. So we can assume without loss of generality that $e_1^i, \ldots, e_n^i$ is this standard basis. Consider then

$$\Big| \vec{n}(\Psi_i(s)) \cdot (\Psi_i(t) - \Psi_i(s))) \Big| = \Big| (\nabla_s g_i(s)\,,\, -1) \cdot (t - s\,,\, g_i(t) - g_i(s)) \Big|$$
$$= \Big| g_i(s) - g_i(t) - \nabla_s g_i(s) \cdot (s - t) \Big| \le c\, |s - t|^{1+\alpha}$$

Thus, we get

$$w(\Psi_i(t), \Psi_i(s)) \le \frac{c}{|s - t|^\nu} \tag{4.23}$$

with $\nu = n + 1 - 2(1+\alpha)$.

Now, we have also to consider the sets $\{(t,s) \in \mathcal{O}_j \times \mathcal{O}_i \,:\, |\Psi_j(t) - \Psi_i(s)| < \mu_0\}$ with $i \neq j$ in the case $\Sigma \cap U_i \cap U_j \neq \emptyset$. At the expense of some technical complications, we obtain an estimate similar to (4.23).

We choose

$$\mu_0 = \frac{1}{2} \inf_{i,j=1,\ldots,m} \sup_{y,z \in \Sigma \cap U_i \cap U_j} |z - y| \,.$$

Then if $|\Psi_j(t) - \Psi_i(s)| \leq \mu_0$ for some $t \in \mathcal{O}_j$ and $s \in \mathcal{O}_i$, there must exist a unique $\tilde{s}_t \in \mathcal{O}_i$ (resp. $\tilde{t_s} \in \mathcal{O}_j$), such that $\Psi_j(t) = \Psi_i(\tilde{s}_t)$ (resp. $\Psi_i(s) = \Psi_j(\tilde{t_s})$).

For the sake of notational simplicity, we shall assume in the following that $|\Psi_j(t) - \Psi_i(s)| \leq \mu_0$ implies the representation $\Psi_j(t) = \Psi_i(\tilde{s}_t)$ for a $\tilde{s}_t \in \mathcal{O}_i$.

We then have, from the fact that Ψ_i, Ψ_i^{-1} are $\mathcal{C}^{1,\alpha}$-bijections, the relations

$$c_1 \, |\Psi_j(t) - \Psi_i(s)| \leq |s - \tilde{s}_t| \leq c_2 \, |\Psi_j(t) - \Psi_i(s)| \,, \tag{4.24}$$

with positive constants c_1, c_2.

In the sets $\{(t,s) \in \mathcal{O}_j \times \mathcal{O}_i \,:\, |\Psi_j(t) - \Psi_i(s)| < \mu_0\}$, we write $\Psi_j(t) = \Psi_i(\tilde{s}_t)$. Then, arguing as above, we obtain the inequality

$$w(\Psi_j(t), \Psi_i(s)) \leq \frac{c}{|s - \tilde{s}_t|^{\nu}} \,,$$

with $\nu = n + 1 - 2(1+\alpha) < n - 1$.

Third step The operator $\tilde{K}_{i,j}$ maps bounded sets in $L^p(\mathbb{R}^{n-1})$ into relatively compact sets in this space.

Let $\| \, f \, \|_{L^p(\mathbb{R}^{n-1})} \leq M$ and $h \in \mathbb{R}^{n-1}$ with arbitrary small norm. It holds that

$$\begin{aligned}
&\left| \Big(\tilde{K}_{i,j}(f)\Big)(t+h) - \Big(\tilde{K}_{i,j}(f)\Big)(t) \right|^p = \left| \int_{\mathbb{R}^{n-1}} \big[k_{i,j}(t+h,s) - k_{i,j}(t,s)\big] \, f(s) \, ds \right|^p \\
&\leq \left(\int_{\mathbb{R}^{n-1}} |k_{i,j}(t+h,s) - k_{i,j}(t,s)| \right)^{p/p'} \left(\int_{\mathbb{R}^{n-1}} |k_{i,j}(t+h,s) - k_{i,j}(t,s)| \, |f(s)|^p \, ds \right) .
\end{aligned}$$

We want to estimate the first of these integrals. For $\mu < \mu_0$, we introduce the notation

$$\begin{aligned}
\text{for } t \in \mathcal{O}_j \,, \quad A_{i,j}(t;\mu) &:= \{s \in \mathcal{O}_i \,:\, |\Psi_j(t) - \Psi_i(s)| \geq \mu\} \,, \\
\mathcal{A}_{i,j}(\mu) &:= \{(t,s) \in \mathcal{O}_j \times \mathcal{O}_i \,:\, |\Psi_j(t) - \Psi_i(s)| \geq \mu\} \,.
\end{aligned}$$

We note that the sets $\mathcal{A}_{i,j}(\mu)$ are compact. Since for $t \in A_{i,j}(t;\mu)$,

$$\big|\Psi_j(t+h) - \Psi_i(s)\big| \;\; \geq \;\; \big|\Psi_j(t) - \Psi_i(s)\big| - \big|\Psi_j(t+h) - \Psi_j(t)\big| \geq \mu - c\,|h| \,,$$

we have, for $|h|$ sufficiently small, uniformly in t the inclusion

$$A_{i,j}(t;\mu) \subseteq A_{i,j}(t+h;\mu/2) \,.$$

For $i = j$, we simply introduce

$$\text{for } t \in \mathcal{O}_j\,, \quad B_{i,j}(t;\mu) := \{ s \in \mathcal{O}_i \,:\, |t - s| < \mu \}\,,$$

and if $\Sigma \cap U_i \cap U_j \neq \emptyset$ with $i \neq j$, we introduce for $\mu < \mu_0$

$$\text{for } t \in \mathcal{O}_j\,, \quad B_{i,j}(t;\mu) := \{ s \in \mathcal{O}_i \,:\, |\tilde{s}_t - s| < c_2\, \mu \}\,,$$

where c_2 is the constant that appears in (4.24), and $\tilde{s}_t$ is constructed according to the second step.

For $\mu < \mu_0$ we then have the inclusion

$$\mathcal{O}_i \setminus A_{i,j}(t;\mu) \subset B_{i,j}(t;\mu)\,,$$

which follows from (4.24). In addition, we can write, for the same reason,

$$B_{i,j}(t;\mu) \subset B_{i,j}(t+h; 2\,\mu)\,.$$

uniformly in t, for all $|h|$ sufficiently small. It follows that

$$\begin{aligned}
&\int_{\mathbb{R}^{n-1}} |k_{i,j}(t+h,s) - k_{i,j}(t,s)|\, ds = \int_{\mathcal{O}_i} |k_{i,j}(t+h,s) - k_{i,j}(t,s)|\, ds \\
&= \int_{A_{i,j}(t;\mu)} |k_{i,j}(t+h,s) - k_{i,j}(t,s)|\, ds + \int_{\mathcal{O}_i \setminus A_{i,j}(t;\mu)} |k_{i,j}(t+h,s) - k_{i,j}(t,s)|\, ds \\
&\leq \lambda_{n-1}(\mathcal{O}_i) \max_{s \in A_{i,j}(t;\mu)} |k_{i,j}(t+h,s) - k_{i,j}(t,s)| + \int_{\mathcal{O}_i \setminus A_{i,j}(t;\mu)} |k_{i,j}(t+h,s) - k_{i,j}(t,s)|\, ds\,.
\end{aligned}$$

On the one hand,

$$\begin{aligned}
&\max_{s \in A_{i,j}(t;\mu)} |k_{i,j}(t+h,s) - k_{i,j}(t,s)| \\
&\qquad \leq \max_{\substack{(t_1,s_1),\ (t_2,s_2) \in \mathcal{A}_{i,j}(\mu/2) \\ |t_1 - t_2| + |s_1 - s_2| < |h|}} |k_{i,j}(t_1,s_1) - k_{i,j}(t_2,s_2)|\,.
\end{aligned}$$

Since the function $k_{i,j}$ is continuous and bounded on $\mathcal{A}_{i,j}(\mu/2)$, the last expression will tend to zero if $|h| \to 0$.

On the other hand, we have, if $\mu < \mu_0$,

$$\begin{aligned}
&\int_{\mathcal{O}_i \setminus A_{i,j}(t;\mu)} |k_{i,j}(t+h,s) - k_{i,j}(t,s)|\, ds \leq \int_{\mathcal{O}_i \setminus A_{i,j}(t;\mu)} |k_{i,j}(t+h,s)| + |k_{i,j}(t,s)|\, ds \\
&\leq \int_{\mathcal{O}_i \setminus A_{i,j}(t;\mu)} \frac{C}{|s - \tilde{s}_t|^\nu} + \frac{C}{|s - \tilde{s}_{t+h}|^\nu} \leq 2 \int_{B_{i,j}(t;2\mu)} \frac{C}{|s - \tilde{s}_t|^\nu} ds \leq \bar{C} \int_{B_{2\,c_2\mu}(0)} |v|^{-\nu} dv \\
&= C^*\, \mu^{n-1-\nu}\,.
\end{aligned}$$

Thus, given any $\epsilon > 0$ we can choose μ such that for all $|h|$ sufficiently small, the inequality

$$\int_{\mathbb{R}^{n-1}} |k_{i,j}(t+h,s) - k_{i,j}(t,s)|\, ds \leq \epsilon$$

is satisfied. From this fact, it follows that

$$\int_{\mathbb{R}^{n-1}} \left|\Big(\tilde{K}_{i,j}(f)\Big)(t+h) - \Big(\tilde{K}_{i,j}(f)\Big)(t)\right|^p dt$$
$$\leq \epsilon^{p/p'} \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}^{n-1}} |k_{i,j}(t+h,s) - k_{i,j}(t,s)|\, |f(s)|^p\, ds\right) dt$$
$$\leq \epsilon^{p/p'}\, 2 \int_{\mathbb{R}^{n-1}} |f(s)|^p\, ds \leq \epsilon^{p/p'}\, 2\, M\,.$$

Now the Fréchet-Kolmogorov theorem (see Alt [1985]) proves the claim.

Fourth step Suppose that $f_k \rightharpoonup f$ in $L^p(\Sigma)$. We then have the weak convergence $(f_k\,\eta_i^{1/p})(\Psi_i) \rightharpoonup (f\,\eta_i^{1/p})(\Psi_i)$ in $L^p(\mathbb{R}^{n-1})$.

We use the abbreviations $|g_i'(s)| := \sqrt{1+|\nabla g_i(s)|^2}$, and $\tilde{\eta}_j := \eta_j(\Psi_j)$. Now, consider

$$\int_\Sigma \big|K(f_k - f)\big|^p =$$
$$\sum_{j=1}^m \int_{\mathbb{R}^{n-1}} \tilde{\eta}_j(t)\, |g_j'(t)| \left|\sum_{i=1}^m \int_{\mathbb{R}^{n-1}} \tilde{\eta}_i(s)\, |g_i'(s)|\, w(\Psi_j(t),\Psi_i(s))\, \big(f_k - f\big)(\Psi_i(s))\, ds\right|^p dt$$
$$\leq C \sum_{j=1}^m \int_{\mathbb{R}^{n-1}} \left|\sum_{i=1}^m \int_{\mathbb{R}^{n-1}} \tilde{\eta}_j^{1/p}(t)\, \tilde{\eta}_i(s)\, w(\Psi_j(t),\Psi_i(s))\, (f_k - f)(\Psi_i(s))\, ds\right|^p dt$$
$$= C \sum_{j=1}^m \int_{\mathbb{R}^{n-1}} \left|\sum_{i=1}^m \tilde{K}_{i,j}\Big[\big((f_k - f)\,\eta_i^{1/p}\big)(\Psi_i)\Big]\right|^p$$
$$\leq \bar{C} \sum_{i,j=1}^m \int_{\mathbb{R}^{n-1}} \left|\tilde{K}_{i,j}\Big[\big((f_k - f)\,\eta_i^{1/p}\big)(\Psi_i)\Big]\right|^p .$$

According to the third step, for some subsequence, we get

$$\int_\Sigma \big|K(f_k - f)\big|^p \to 0\,.$$

which proves the claim. □

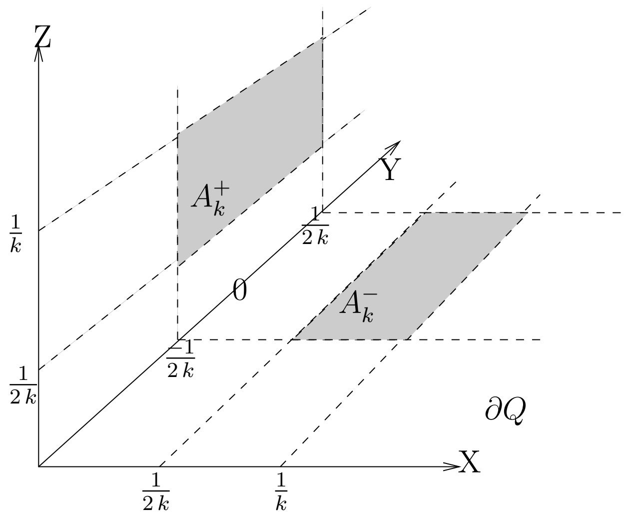
Lemma 4.2.2. Let $\Sigma \in \mathcal{C}^{1,\alpha}$ piecewise have at least one edge. For $1 < p \leq \infty$, we then have $K \notin \mathcal{K}(L^p(\Sigma), L^p(\Sigma))$.

Proof. It is sufficient to consider a counterexample. Let $\Sigma = \partial Q$, where

$$Q := [0,2] \times [-1,1] \times [0,2]\,.$$

For $k \in \mathbb{N}$, define two surfaces $A_k^+,\, A_k^- \subset \Sigma$

$$A_k^+ := \{0\} \times [-1/2\,k, 1/2\,k] \times [1/2\,k, 1/k], \quad A_k^- := [1/2\,k, 1/k] \times [-1/2\,k, 1/2\,k] \times \{0\}\,.$$



Observe that $\mathrm{meas}(A_k^{+,-}) = 1/2\,k^2$. We now consider pairs of points $(z, y) \in A_k^+ \times A_k^-$ (Figure 4.2).

We immediately see that

$$\vec{n}(z)\cdot(y-z) = -(y_1 - z_1) = -y_1\,,\quad \vec{n}(y)\cdot(z-y) = -(z_3 - y_3) = -z_3\,.$$

Observe that by construction

$$\inf_{z\in A_k^+,\, y\in A_k^-} |z_1 - y_1| \geq \frac{1}{2\,k},\quad \inf_{z\in A_k^+,\, y\in A_k^-} |z_3 - y_3| \geq \frac{1}{2\,k}\,.$$

By construction as well,

$$\sup_{z\in A_k^+,\, y\in A_k^-} |z-y| \leq \frac{\sqrt{3}}{k}\,.$$

Therefore, for pairs $(z, y) \in A_k^+ \times A_k^-$, we have

$$w(z,\,y) := \frac{\vec{n}(z)\cdot(y-z)\;\vec{n}(y)\cdot(z-y)}{\pi\,|z-y|^4} = \frac{|y_1 - z_1|\;|z_3 - y_3|}{\pi\,|z-y|^4} \geq \frac{k^2}{36\,\pi}\,. \tag{4.25}$$

Now, for $1 < p < \infty$, consider the functions $f_k(y) := k^{2/p}\,\chi_{A_k^-}(y)$. One readily verifies that

$$\|f_k\|^p_{L^p(\Sigma)} = k^2\;\mathrm{meas}(A_k^-) = \frac{1}{2}\,.$$

It is also obvious that $f_k(y) \to 0$ for all $y \in \Sigma$ as $k \to \infty$. Therefore, $f_k(y) \rightharpoonup 0$ in $L^p(\Sigma)$.

For $z \in A_k^+$, we can in view of (4.25) write that

$$(K(f_k))(z) := \int_\Sigma w(z,\,y)\,f_k(y)\,dy = k^{2/p}\int_{A_k^-} w(z,\,y) \geq \frac{k^{2/p}\,k^2\;\mathrm{meas}(A_k^-)}{36\,\pi} = \frac{k^{2/p}}{72\,\pi}\,.$$

Thus, for $z \in A_k^+$

$$|(K(f_k))(z)|^p \geq \frac{k^2}{72^p\,\pi^p}\,.$$

We conclude that

$$\int_{\Sigma} |(K(f_k))(z)|^p \, dz \geq \int_{A_k^+} |(K(f_k))(z)|^p \, dz \geq \frac{k^2}{72^p \, \pi^p} \, \text{meas}(A_k^+) = \frac{1}{2 \; 72^p \, \pi^p} > 0 \, .$$

This shows that there exists no subsequence such that $K(f_{k_j}) \to 0$ in $L^p(\Sigma)$, proving the claim. $\square$

We give the technical proof of Lemma 4.1.11

Lemma 4.2.3. Assume that $\Sigma \in \mathcal{C}^{1,\alpha}$. For $z_1, \, z_2 \in \Sigma$, define $A(z_1, \, z_2) := \{y \in \Sigma : \Theta(z_1, \, y) \neq \Theta(z_2, \, y)\}$. Then, we have

$$\max_{z_1, \, z_2 \in \Sigma \, : \, |z_1 - z_2| \leq \delta} \text{meas}(A(z_1, \, z_2)) \leq c \, \delta^\alpha \, .$$

Proof. We denote by C^* the $\mathcal{C}^{1,\alpha}-$norm of the surface Σ.

Since $\Sigma \in \mathcal{C}^{1,\alpha}$, the set of the obstacles $\text{conv}(\Omega_0) \cap \Omega_{\text{op}}$ consists of finitely many disjoint bounded domains $O_1, \, \ldots, O_r$. Without loss of generality, we consider only one of these obstacles that we denote by O.

We consider two arbitrary $z_1, \, z_2 \in \Sigma$ such that $|z_1 - z_2| \leq \delta$. We choose a positive number $\delta < \inf_{i,j=1,\ldots,m \, : \, i \neq j} \text{dist}(\overline{\Omega_i}, \, \overline{\Omega_j})$. Then, there exists some $i_0 \in \{1, \ldots, m\}$ such that the points $z_1, \, z_2$ belong to the boundary of the same domain Ω_{i_0}.

We now want to construct a particular covering of the set $A(z_1, \, z_2)$ in order to estimate its measure. For $\phi \in [0, \, \pi]$, denote by $P(z_1, \, z_2, \, \phi)$ the plane containing $z_1, \, z_2$ and crossing the $(x, \, y)$-plane at angle ϕ. We take the intersection line between the two plane to be the $X-$axis of a new coordinate system. By Fubini's theorem, we obviously can write

$$\text{meas}(A(z_1, \, z_2)) = \int_0^\pi \text{meas}(A(z_1, \, z_2) \cap P(z_1, \, z_2, \, \phi)) \, d\phi \, . \tag{4.26}$$

The intersection of the plane $P(z_1, \, z_2, \, \phi)$ with the surface Σ consists of two-dimensional domains, whose $C^{1,\alpha}-$constant is bounded by C^*. We denote by O' the intersection $O \cap P(z_1, \, z_2, \, \phi)$, and by Ω'_{i_0} the intersection $\Omega_{i_0} \cap P(z_1, \, z_2, \, \phi)$.

For simplicity, we consider only the case that $z_1, \, z_2 \in \partial O$, as depicted in Figure 4.2.

By assumption, we can find an interval I and a function $g \in \mathcal{C}^{1,\alpha}(I)$ such that $\partial \mathcal{O}' = (x, \, g(x))$, such that $g \geq 0$ on I and such that $\|g\|_{C^{1,\alpha}(I)} \leq C^*$.

For the angle θ between the two tangents crossing $\partial O'$ at z_i for $i = 1, \, 2$, we have

$$\theta = \arctan \left| \frac{g'(x_1) - g'(x_2)}{1 + g'(x_1) \, g'(x_2)} \right| \, . \tag{4.27}$$

We can parameterize

$$A(z_1, \, z_2) \cap P(z_1, \, z_2, \, \phi) := \{\tilde{z} + r(\psi) \, (\cos \psi, \sin(\psi)) \, : \, \psi \in [0, \, \theta]\} \, ,$$

where $\tilde{z}$ is the intersection point of the two tangents, with a radius function r whose derivative can be estimated in terms of the maximum of the derivative of the surface Σ.

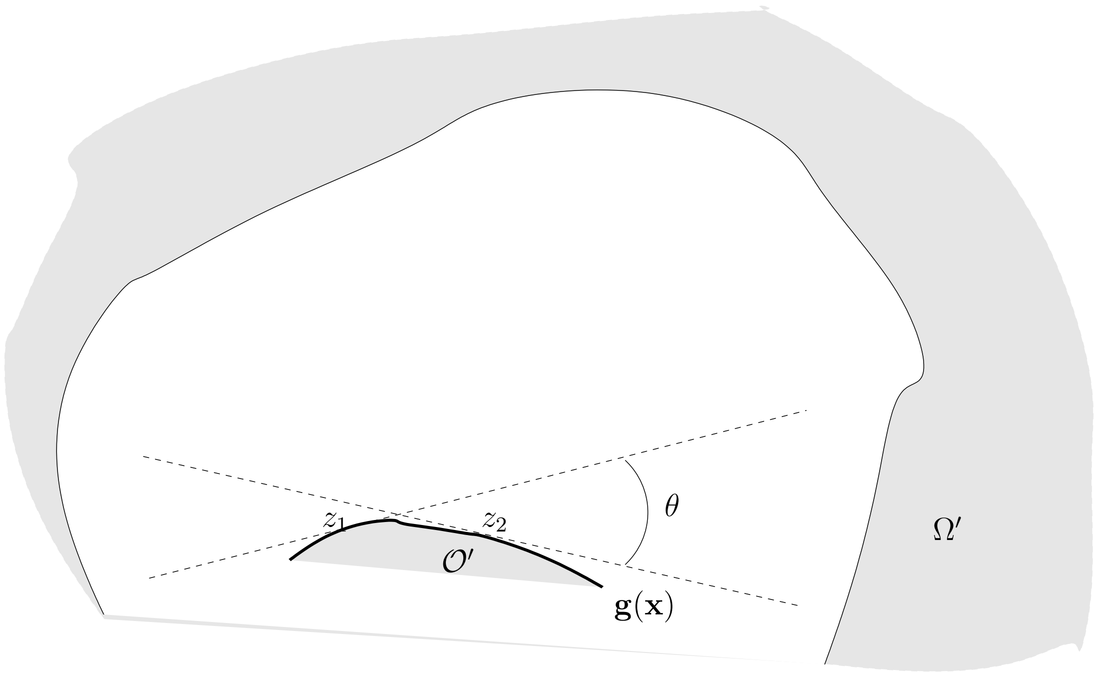


Figure 4.3: The domain $\Omega \cap P(z_1,\, z_2,\, \phi)$

The function r is itself uniformly bounded, for example by the diameter of the transparent cavity Ω_0. We further have

$$\operatorname{meas}(A(z_1,\, z_2) \cap P(z_1,\, z_2,\, \phi)) = \int_0^{\theta} \sqrt{r^2(\psi) + r'^2(\psi)}\, d\psi \le \tilde{c}\, |\theta|\,. \tag{4.28}$$

It remains to estimate θ. Since the function g' belongs to $C^{1,\alpha}$, it is obvious that

$$\theta \le C^*\, |x_1 - x_2|^{\alpha} \le C^*\, \delta^{\alpha}\,.$$

Using the formula (4.26) and (4.28), we finally prove the claim. □

4.3 Coercivity inequalities involving the operators of heat radiation

In this section we want to study the coercivity of the nonlinear operator

$$\big\langle A\,\theta,\, \psi\big\rangle + \int_{\Sigma} G(\sigma\, |\theta|^3\, \theta)\, \psi\,.$$

Here, the symbol $\langle \cdot, \cdot \rangle$ denotes the duality product between a suitable Banach space and its dual, and the operator A is simply defined as

$$\big\langle A\,\theta,\, \psi\big\rangle := \int_{\Omega} \kappa(\theta)\, \nabla\theta \cdot \nabla\psi\,.\,.$$

A more general operator A has been treated in Druet [2009a]. As was shown in Tiihonen [1997b], one easily obtains coercivity on the space $V^{2,5}(\Omega)$ (cp. (3.61)) if the domain Ω is not an enclosure. In the latter case, in view of Lemma 4.1.5, $\| \mathbf{H} \|_{\mathcal{L}(L^{5/4}(\Sigma),L^{5/4}(\Sigma))} < 1$, and one has

$$\int_\Sigma G(\sigma\,|\theta|^3\,\theta)\,\theta \geq (1- \| \mathbf{H} \|_{\mathcal{L}(L^{5/4}(\Sigma),L^{5/4}(\Sigma))}) \int_\Sigma |\theta|^5\,,$$

(see also Lemma 4.1.8, (3) above). For the case that Ω might be an enclosure, the following Lemma states a first general coercivity result.

Lemma 4.3.1. Assume that $\Sigma \in \mathcal{C}^1$ piecewise. Let $r, s > 0$ be two numbers such that $r + s < 4$. Then there exists a constant $c = c_{r,s} > 0$ such that for all $\psi \in W^{1,2}_\Gamma(\Omega)$

$$\langle A\,\psi,\,\psi\rangle + \int_\Sigma G(|\psi|^{r-1}\,\psi)\,|\psi|^{s-1}\,\psi \geq c\,\min\left\{\, \| \psi \|^2_{W^{1,2}_\Gamma(\Omega)},\, \| \psi \|^{r+s}_{W^{1,2}_\Gamma(\Omega)}\,\right\}.$$

Proof. We at first show that there exists a constant $\bar{c} > 0$ such that

$$\langle A\,\psi,\,\psi\rangle + \int_\Sigma G(|\psi|^{r-1}\,\psi)\,|\psi|^{s-1}\,\psi \geq \bar{c}\, \| \psi \|^2_{W^{1,2}_\Gamma(\Omega)}\,,$$

for all $\psi \in W^{1,2}_\Gamma(\Omega)$ such that $\| \psi \|_{W^{1,2}_\Gamma(\Omega)} \geq 1$. Suppose that the latter claim is not true. Then we can find a sequence $\{\psi_n\} \subset W^{1,2}_\Gamma(\Omega)$ such that

$$\langle A\,\psi_n,\,\psi_n\rangle + \int_\Sigma G(|\psi_n|^{r-1}\,\psi_n)\,|\psi_n|^{s-1}\,\psi_n \quad\leq\quad \frac{1}{n}\, \| \psi_n \|^2_{W^{1,2}_\Gamma(\Omega)}\;.$$

We set

$$\tilde{\psi}_n := \psi_n/\|\psi_n\|_{W^{1,2}_\Gamma(\Omega)}, \qquad \| \tilde{\psi}_n \|_{W^{1,2}_\Gamma(\Omega)} = 1\,.$$

Thus, passing to subsequences if necessary

$$\tilde{\psi}_n \rightharpoonup \tilde{\psi} \text{ in } W^{1,2}_\Gamma(\Omega)\,, \quad \tilde{\psi}_n \to \tilde{\psi} \text{ almost everywhere on } \Sigma\,.$$

We find that

$$\langle A\,\tilde{\psi}_n,\,\tilde{\psi}_n\rangle + \int_\Sigma G(|\tilde{\psi}_n|^{r-1}\,\tilde{\psi}_n)\,|\tilde{\psi}_n|^{s-1}\,\tilde{\psi}_n \leq \frac{1}{n}\,. \tag{4.29}$$

Since the choice $r + s < 4$ implies that $\frac{4s}{4-r} < 4$, we can again pass to a subsequence to obtain that

$$|\tilde{\psi}_n|^{r-1}\,\tilde{\psi}_n \rightharpoonup |\tilde{\psi}|^{r-1}\,\tilde{\psi} \text{ in } L^{\frac{4}{r}}(\Sigma)\,,$$
$$|\tilde{\psi}_n|^{s-1}\,\tilde{\psi}_n \to |\tilde{\psi}|^{s-1}\,\tilde{\psi} \text{ in } L^{\frac{4}{4-r}}(\Sigma)\,,$$

which allows us to pass to the limit in (4.29). Taking into account Lemma 4.1.8, we now have

$$\tilde{\psi} = c_i \text{ in each } \Omega_i\,, \qquad \tilde{\psi} = c \text{ on } \Sigma, \qquad \tilde{\psi} = 0 \text{ on } \Gamma\,.$$

This leads to $\tilde{\psi} = 0$. As a matter of fact, we can always find a part $\Omega_{i_0} \subset \Omega$ such that both $\partial\Omega_{i_0} \cap \Sigma$ and $\partial\Omega_{i_0} \cap \Gamma$ are not empty. Considering (4.29), we find that $\tilde{\psi}_n \to 0 \in W^{1,2}_\Gamma(\Omega)$, which is a contradiction.

In the case that $\| \psi \|_{W^{1,2}_\Gamma(\Omega)} < 1$, we use an analogous argument replacing $\| \psi_n \|^2_{W^{1,2}_\Gamma(\Omega)}$ by $\| \psi_n \|^{r+s}_{W^{1,2}_\Gamma(\Omega)}$. The claim follows. $\square$

In the case that the operator K is compact, a better coercivity result was proven in Laitinen and Tiihonen [2001].

Lemma 4.3.2. Let $\Sigma \in \mathcal{C}^{1,\alpha}$. Let $r, s > 0$. Then there exists a constant $c = c_{r,s} > 0$ such that for all $\psi \in V_\Gamma^{2,r+s}(\Omega)$,

$$\langle A\,\psi,\,\psi\rangle + \int_\Sigma G(|\psi|^{r-1}\,\psi)\,|\psi|^{s-1}\,\psi \geq c\,\min\left\{\,\|\,\psi\,\|^2_{V_\Gamma^{2,r+s}(\Omega)},\,\|\,\psi\,\|^{r+s}_{V_\Gamma^{2,r+s}(\Omega)}\,\right\}.$$

Proof. See Laitinen and Tiihonen [2001], Theorem 3. □

The inequalities in Lemma 4.3.1 and Lemma 4.3.2 establish coercivity properties of the operator of heat radiation taken in connection with the heat conduction. The next statements show that the radiation operator on smooth surfaces by itself already exerts some coercivity.

Lemma 4.3.3. Let $\Sigma \in \mathcal{C}^{1,\alpha}$. Let $r, s > 0$ be to numbers with $s \leq r+1$. Then the following statements are valid:

(1) There exists a positive constant $c_{r,s}$ such that for all $\psi \in L^{r+1}(\Sigma)$,

$$\int_\Sigma G(|\psi|^{r-1}\,\psi)\,\psi + \left(\int_\Sigma |\psi|^s\right)^{\frac{r+1}{s}} \geq c\,\|\,\psi\,\|^{r+1}_{L^{r+1}(\Sigma)}\;.$$

(2) If the domain Ω is an enclosure, there exists a positive constant $\bar{c}_{r,s}$ such that

$$\int_\Sigma G(|\psi|^{r-1}\,\psi)\,\psi \geq \bar{c}\,\|\,\psi\,\|^{r+1}_{L^{r+1}(\Sigma)}\,,$$

for all $\psi \in L^{r+1}(\Sigma)$ such that $\int_\Sigma \psi\,dS = 0$.

Proof. (1): We assume that the assertion is false, and we seek a contradiction. We can construct a sequence $\{\tilde{\psi}_n\} \subset L^{r+1}(\Sigma)$ such that $\|\,\tilde{\psi}_n\,\|_{L^{r+1}(\Sigma)} = 1$ and

$$\int_\Sigma G(|\tilde{\psi}_n|^{r-1}\,\tilde{\psi}_n)\,\tilde{\psi}_n + \left(\int_\Sigma |\tilde{\psi}_n|^s\right)^{\frac{r+1}{s}} < \frac{1}{n}\,. \tag{4.30}$$

Extracting subsequences, we find that

$$\tilde{\psi}_n \rightharpoonup \tilde{\psi} \text{ in } L^{r+1}(\Sigma)\,, \qquad |\tilde{\psi}_n|^{r-1}\,\tilde{\psi}_n \rightharpoonup w \text{ in } L^{\frac{r+1}{r}}(\Sigma)\,.$$

Passing to the limit in (4.30), we can write

$$\limsup_{n\to\infty} \int_\Sigma \epsilon\,|\tilde{\psi}_n|^{r+1} - \lim_{n\to\infty} \int_\Sigma \epsilon\,\tilde{\mathbf{H}}(|\tilde{\psi}_n|^{r-1}\,\tilde{\psi}_n)\,\tilde{\psi}_n \leq 0\,,$$

and, using the compactness of $\tilde{\mathbf{H}}$ from $L^{1+1/r}(\Sigma)$ into itself, we get

$$\limsup_{n\to\infty} \int_\Sigma \epsilon\,|\tilde{\psi}_n|^{r+1} - \int_\Sigma \epsilon\,\tilde{\mathbf{H}}(w)\,\tilde{\psi} \leq 0\,.$$

On the other hand, we have by the same tools that

$$\int_\Sigma \epsilon\, \tilde{\mathbf{H}}(w)\, \tilde\psi = \liminf_{n\to\infty} \int_\Sigma \epsilon\, \tilde{\mathbf{H}}(|\tilde\psi_n|^{r-1}\, \tilde\psi_n)\, \tilde\psi$$
$$\le \liminf_{n\to\infty} \left\| \epsilon^{\frac{r}{r+1}}\, \tilde{\mathbf{H}}(|\tilde\psi_n|^{r-1}\, \tilde\psi_n) \right\|_{L^{1+1/r}(\Sigma)} \ \| \, \epsilon^{\frac{1}{r+1}}\, \tilde\psi \, \|_{L^{r+1}(\Sigma)} \ . \tag{4.31}$$

In view of Lemma 4.1.14 we can write

$$\left\| \epsilon^{\frac{r}{r+1}}\, \tilde{\mathbf{H}}(|\tilde\psi_n|^{r-1}\, \tilde\psi_n) \right\|_{L^{1+1/r}(\Sigma)} = \left\| \tilde{\mathbf{H}}_{\frac{r+1}{r}}\, (\epsilon^{\frac{r}{r+1}}\, |\tilde\psi_n|^{r-1}\, \tilde\psi_n) \right\|_{L^{1+1/r}(\Sigma)}$$
$$\le \| \, \epsilon^{\frac{r}{r+1}}\, |\tilde\psi_n|^r \, \|_{L^{1+1/r}(\Sigma)} = \left(\int_\Sigma \epsilon\, |\tilde\psi_n|^{r+1} \right)^{\frac{r}{r+1}} \ .$$

Thus, we can continue the estimate (4.31) by

$$\int_\Sigma \epsilon\, \tilde{\mathbf{H}}(w)\, \tilde\psi \quad \le \quad \| \, \epsilon^{\frac{1}{r+1}}\, \tilde\psi \, \|_{L^{r+1}(\Sigma)} \ \liminf_{n\to\infty} \left(\int_\Sigma \epsilon\, |\tilde\psi_n|^{r+1} \right)^{\frac{r}{r+1}}$$

It follows that

$$\limsup_{n\to\infty} \int_\Sigma \epsilon\, |\tilde\psi_n|^{r+1} \le \| \, \epsilon^{\frac{1}{r+1}}\, \tilde\psi \, \|_{L^{r+1}(\Sigma)} \ \left(\limsup_{n\to\infty} \int_\Sigma \epsilon\, |\tilde\psi_n|^{r+1} \right)^{\frac{r}{r+1}} \ ,$$

which implies that $\limsup_{n\to\infty} \| \, \epsilon^{\frac{1}{r+1}}\, \tilde\psi_n \, \|^{r+1}_{L^{r+1}(\Sigma)} \le \| \, \epsilon^{\frac{1}{r+1}}\, \tilde\psi \, \|^{r+1}_{L^{r+1}(\Sigma)}$. Combining this with the usual lower semicontinuity of the norm, we obtain for a subsequence that

$$\lim_{n\to\infty} \| \, \epsilon^{\frac{1}{r+1}}\, \tilde\psi_n \, \|^{r+1}_{L^{r+1}(\Sigma)} = \| \, \epsilon^{\frac{1}{r+1}}\, \tilde\psi \, \|^{r+1}_{L^{r+1}(\Sigma)} \, ,$$

which, in its turn, yields

$$\tilde\psi_n \to \tilde\psi \text{ in } L^{r+1}(\Sigma) \, . \tag{4.32}$$

Reconsidering (4.30) for this subsequence, we now obtain that

$$\int_\Sigma G(|\tilde\psi|^{r-1}\, \tilde\psi)\, \tilde\psi = 0 \, . \tag{4.33}$$

By Lemma 4.1.8, it follows that $\tilde\psi$ is constant. But since $s \le r+1$, (4.30) also gives that $\left(\int_\Sigma |\tilde\psi|^s \right)^{\frac{r+1}{s}} = 0$. Thus, $\psi \equiv 0$ on Σ, a contradiction by the strong convergence (4.32).

(2): We prove the second estimate by the same arguments, obtaining the consequence (4.33). Then, by the strong convergence (4.32), we find that $\tilde\psi$ has mean value zero on Σ. We can finish the proof analogously. □

We now prove a last coercivity result, which will in particular help us to produce estimates in the case that f belongs only to L^1.

Lemma 4.3.4. Let $\Sigma \in \mathcal{C}^{1,\alpha}$. Then there exists a positive constant c such that

$$\int_\Sigma G(\psi)\,\mathrm{sign}(\psi) \geq c \parallel \psi \parallel_{L^1(\Sigma)},$$

for all $\psi \in L^1(\Sigma)$ such that $\int_\Sigma \psi\, dS = 0$.

Proof. Again, suppose that the claim is not true. Then, it is possible to construct a sequence $\{\psi_n\} \subset L^1(\Sigma)$ with the properties

$$\parallel \psi_n \parallel_{L^1(\Sigma)} = 1\,, \qquad \int_\Sigma \psi_n = 0\,, \qquad \int_\Sigma G(\psi_n)\,\mathrm{sign}(\psi_n) \leq \frac{1}{n}\,.$$

Now, since $\psi_n\, G(\mathrm{sign}(\psi_n)) = |\psi_n| - \psi_n\, \mathbf{H}(\mathrm{sign}(\psi_n)) \geq 0$, and using also the fact that G is selfadjoint, we can write that

$$\begin{aligned}\frac{1}{n} \geq \int_\Sigma G(\psi_n)\,\mathrm{sign}(\psi_n) = \int_\Sigma \psi_n\, G(\mathrm{sign}(\psi_n)) &= \int_\Sigma |\psi_n|\, |G(\mathrm{sign}(\psi_n))| \\ &= \int_\Sigma \epsilon\, |\psi_n|\, |\mathrm{sign}(\psi_n) - \tilde{\mathbf{H}}(\mathrm{sign}(\psi_n))|\,. \end{aligned} \tag{4.34}$$

Choosing a $q > \frac{1}{\alpha}$, we can find a subsequence $\mathrm{sign}(\psi_n) \rightharpoonup u \in L^q(\Sigma)$. We have, in particular, that $|u| \leq 1$ almost everywhere on Σ. By Lemma 4.1.10, we can again pass to a subsequence if necessary to find that

$$\tilde{\mathbf{H}}(\mathrm{sign}(\psi_n)) \longrightarrow \tilde{\mathbf{H}}(u) \qquad \text{in } C(\Sigma)\,. \tag{4.35}$$

We distinguish two cases.

For the first case, we assume that $u = 1$ almost everywhere on Σ. By the uniform convergence of $\{\tilde{\mathbf{H}}(\mathrm{sign}(\psi_n))\}$, and by (4.34), we obtain that

$$\begin{aligned}\lim_{n\to\infty} \int_\Sigma |\psi_n| &= \lim_{n\to\infty} \int_\Sigma |\psi_n| - \psi_n = \lim_{n\to\infty} \int_\Sigma |\psi_n|\, |\mathrm{sign}(\psi_n) - 1| \\ &= \lim_{n\to\infty} \int_\Sigma |\psi_n|\, |\mathrm{sign}(\psi_n) - \tilde{\mathbf{H}}(u)| = \lim_{n\to\infty} \int_\Sigma |\psi_n|\, |\mathrm{sign}(\psi_n) - \tilde{\mathbf{H}}(\mathrm{sign}(\psi_n))| = 0\,.\end{aligned}$$

This is a contradiction. We argue analogously if $u = -1$ almost everywhere on Σ.

Thus, we must have the second case $u \not\equiv 1, -1$. In this case we know, thanks to Lemma 4.1.14, that $|\tilde{\mathbf{H}}(u)| < 1$ on Σ. This implies, by the continuity of $\tilde{\mathbf{H}}(u)$, that $1 > \max_\Sigma |\tilde{\mathbf{H}}(u)| =: \gamma_0$. We have

$$\begin{aligned}0 = \lim_{n\to\infty} \int_\Sigma \epsilon\, |\psi_n|\, |\mathrm{sign}(\psi_n) - \tilde{\mathbf{H}}(\mathrm{sign}(\psi_n))| &= \lim_{n\to\infty} \int_\Sigma \epsilon\, |\psi_n|\, |\mathrm{sign}(\psi_n) - \tilde{\mathbf{H}}(u)| \\ &\geq \epsilon_0\, (1 - \gamma_0)\, \lim_{n\to\infty} \int_\Sigma |\psi_n|\,.\end{aligned}$$

□

Remark 4.3.5. For $p \in [1, \infty[$, define $L^p_M(\Sigma) := \{f \in L^p(\Sigma) \, : \, \int_\Sigma f \, dS = 0\}$. Then, if (3.1) is satisfied $G^{-1} \in \mathcal{L}(L^p_M(\Sigma), L^p_M(\Sigma))$. As a matter of fact, in view of Lemma 4.1.14, the invertibility of G is equivalent to the unique solvability in $L^p_M(\Sigma)$ of the equation

$$(I - \tilde{\mathbf{H}})(f) = \frac{g}{\epsilon}\,,$$

for any given $g \in L^p_M(\Sigma)$. Since $I - \tilde{\mathbf{H}}$ vanishes only for constants, $I - \tilde{\mathbf{H}}$ is injective from $L^p_M(\Sigma)$ into itself. In view of Lemma 4.1.14, (1), the claim follows.

The following Lemma is useful if we want to use test functions that depend non linearly on temperature. It generalizes properties proved in Laitinen and Tiihonen [2001], Meyer [2006].

Lemma 4.3.6. Let Ω be an enclosure. Let $F : \mathbb{R} \to \mathbb{R}$ be a nondecreasing, continuous function with $F(0) = 0$ and $|F(t)| \leq C_0\,(1+|t|^s)$ as $|t| \to \infty$ $(0 \leq s < \infty)$. Let $0 \leq r < \infty$ be an arbitrary number. Then for all $\psi \in L^{r+s}(\Sigma)$,

$$\int_\Sigma G(|\psi|^{r-1}\,\psi)\,F(\psi) \geq 0\,.$$

Proof. We fix $n \in \mathbb{N}$. For $i = 1, 2, \ldots$, we define

$$a_i^{(n)} := F\left(\frac{i}{n}\right) - F\left(\frac{i-1}{n}\right)\,, \qquad a_{-i}^{(n)} := F\left(\frac{-i-1}{n}\right) - F\left(\frac{-i}{n}\right)\,.$$

Since F is nondecreasing, we obviously have $a_i^{(n)} \geq 0$ and $a_{-i}^{(n)} \leq 0$. Denoting by $\phi_{[a,b]}$ the characteristic function of the interval $[a,b]$, we introduce

$$F_n(t) := \sum_{i=1}^{\infty} a_i^{(n)}\,\phi_{[i/n,+\infty[}(t) + a_{-i}^{(n)}\,\phi_{]-\infty,-i/n]}(t)\,.$$

We can write

$$\int_\Sigma G(|\psi|^{r-1}\,\psi)\,F_n(\psi)$$
$$= \sum_{i=1}^{\infty}\left\{a_i^{(n)}\int_\Sigma G(|\psi|^{r-1}\,\psi)\,\phi_{[i/n,+\infty[}(\psi) + a_{-i}^{(n)}\int_\Sigma G(|\psi|^{r-1}\,\psi)\,\phi_{]-\infty,-i/n]}(\psi)\right\}\,.$$

Now, since Ω is an enclosure, $G(1) = 0$, and we have

$$\begin{aligned}\int_\Sigma G(|\psi|^{r-1}\,\psi)\,\phi_{[i/n,+\infty[}(\psi) &= \int_\Sigma G\left(|\psi|^{r-1}\,\psi - \frac{i^r}{n^r}\right)\,\phi_{[i/n,+\infty[}(\psi)\\ &= \int_\Sigma \left(|\psi|^{r-1}\,\psi - \frac{i^r}{n^r}\right)\,G(\phi_{[i/n,+\infty[}(\psi))\,.\end{aligned}$$

As usual, we observe that

$$G(\phi_{[i/n,+\infty[}(\psi)) = \begin{cases} 1 - \mathbf{H}(\phi_{[i/n,+\infty[}(\psi)) & \geq 0 \text{ if } \psi \geq i/n\,,\\ -\mathbf{H}(\phi_{[i/n,+\infty[}(\psi)) & \leq 0 \text{ if } \psi < i/n\,.\end{cases}$$

This means that $sign\Big((|\psi|^{r-1}\,\psi - \frac{i^r}{n^r})\,G(\phi_{[i/n,+\infty[}(\psi))\Big) = 1$, whence

$$a_i^{(n)} \int_\Sigma G(|\psi|^{r-1}\,\psi)\,\phi_{[i/n,\infty[}(\psi) \geq 0\,,$$

for all $i = 1, 2, \dots$. In the same way we show that $a_{-i}^{(n)} \int_\Sigma G(|\psi|^{r-1}\,\psi)\,\phi_{]-\infty,-i/n]}(\psi) \geq 0$. We thus proved that

$$\int_\Sigma G(|\psi|^{r-1}\,\psi)\,F_n(\psi) \geq 0\,. \tag{4.36}$$

Observe that for any $t \in \mathbb{R}^+$, we can find $i_0^{(n)} \in \mathbb{N}$ such that $t \in \left[\frac{i_0^{(n)}}{n}\,,\,\frac{i_0^{(n)}+1}{n}\right[$. We have

$$\begin{aligned} F(t) - F_n(t) &= F(t) - \sum_{i=1}^{i_0} a_i^{(n)}\,\phi_{[i/n,+\infty[}(t) = F(t) - \sum_{i=1}^{i_0^{(n)}} F\left(\frac{i}{n}\right) - F\left(\frac{i-1}{n}\right) \\ &= F(t) - F\left(\frac{i_0^{(n)}}{n}\right) \to 0 \qquad \text{as } n \to \infty\,, \end{aligned}$$

which is true for all $t \in \mathbb{R}$. By an analogous consideration for $t \in \mathbb{R}^-$, we easily obtain that $F_n(t) \to F(t)$ for all $t \in \mathbb{R}$. We also immediately see that $|F_n(t)| \leq |F(t)|$ for all $t \in \mathbb{R}$. It follows that

$$F_n(\psi) \to F(\psi) \text{ in } L^s(\Sigma) \text{ for all } \psi \in L^s(\Sigma)\,.$$

Passage to the limit as $n \to \infty$ in (4.36) proves the assertion. □

4.4 Passage to the limit in a PDE with nonlocal radiation boundary condition

Solutions to systems with right-hand side in L^1 are usually obtained as asymptotic limit of regularized problems. But passage to the limit with a nonlocal radiation operator is not necessarily a trivial matter[1].

In the paper Druet [2009a] devoted to the stationary heat equation with right-hand side $L^1(\Omega)$ a uniform bound on the sequence $\{\theta_\delta\}$ of approximate solutions could be obtained only in the $L^4(\Sigma)$ norm. This does not give the weak convergence $\theta_\delta^4 \rightharpoonup \theta^4$ in $L^1(\Sigma)$, which is necessary to have

$$\lim_{\delta\to 0} \int_\Sigma G(\sigma\,\theta_\delta^4)\,\xi = \int_\Sigma G(\sigma\,\theta^4)\,\xi\,, \tag{4.37}$$

at least for all $\xi \in L^\infty(\Sigma)$.

In this section, we show that if the approximate solution θ_δ satisfies a certain integral relation, then (4.37) is valid. This auxiliary result will allow to shorten technical parts of the proofs in the next chapters.

[1]Especially in the case of the parabolic problem. The proofs in Metzger [1999], Lemma 4.11 and in Laitinen and Tiihonen [2001], Lemma 7 are not correct.

The elliptic case Assume that for all values of a parmeter $\delta > 0$, there exists $\theta_\delta \in V^{2,5}(\Omega)$ such that, for all $\xi \in V^{2,5}_\Gamma(\Omega)$,

$$\int_\Omega k(\theta_\delta)\, \nabla\theta_\delta \cdot \nabla\xi + \int_\Sigma G(\sigma\, \theta_\delta^4)\, \xi = \int_\Omega \tilde{f}_\delta\, \xi\,, \tag{4.38}$$

where the function $\tilde{f}_\delta \in L^2(\Omega)$ is an approximation of an L^1-force term $\tilde{f}$ such that

$$\tilde{f}_\delta \longrightarrow \tilde{f} \quad \text{in } L^1(\Omega)\,, \tag{4.39}$$

and where $k = k_i \in C(\overline{\Omega_i})$ for $i = 0, \ldots, m$.

Proposition 4.4.1. Let Σ be of class $\mathcal{C}^{1,\alpha}$.

Assume that $\theta_\delta \in V^{2,5}(\Omega)$ satisfies (4.38), that $\theta_\delta \geq 0$ almost everywhere in Ω, and that there exists a constant C independent of δ such that $\|\theta_\delta\|_{L^4(\Sigma)} \leq C$. In addition, let $\theta_\delta \rightharpoonup \theta$ in $W^{1,r}(\Omega)$ for some $1 < r < \infty$.

If (4.39) is valid, we can find a subsequence (not relabelled) such that

$$\theta_\delta \longrightarrow \theta \quad \text{in } L^4(\Sigma)\,,$$

Proof. By Rellich's compactness results, the weak convergence of the sequence $\{\theta_\delta\}$ in $W^{1,r}(\Omega)$ for $r > 1$ implies the existence of a subsequence such that

$$\theta_\delta \longrightarrow \theta \quad \text{almost everywhere in } \Omega \text{ and on } \Sigma\,. \tag{4.40}$$

On the other hand, by Lemma 4.1.14, (4), we find the existence of some $u \in L^1(\Sigma)$ such that

$$\tilde{\mathbf{H}}(\theta_\delta^4) \rightharpoonup u \text{ in } L^1(\Sigma)\,.$$

For an arbitrary $\xi \in C_c^\infty(\Omega)$, we pass to the limit $\delta \to 0$ in the relation (4.38). Considering (4.39), we obtain the existence of $\lim_{\delta\to 0} \int_\Sigma \epsilon\, \sigma\, \theta_\delta^4\, \xi$ and the identity

$$\int_\Omega k(\theta)\, \nabla\theta \cdot \nabla\xi + \lim_{\delta\to 0} \int_\Sigma \epsilon\, \sigma\, \theta_\delta^4\, \xi - \int_\Sigma \epsilon\, \sigma\, u\, \xi = \int_\Omega \tilde{f}\, \xi\,. \tag{4.41}$$

For $\gamma > 0$ and $t \in \mathbb{R}^+$, define

$$g_\gamma(t) := \frac{1}{1 + \gamma\, t^4}\,.$$

Observe that g_γ is continuous on $\mathbb{R}^+$ and monotonely decreasing. In order to compute $\lim_{\delta\to 0} \int_\Sigma G(\sigma\, \theta_\delta^4)\, \xi$, we now test the relation (4.38) with the function $g_\gamma(\theta_\delta)\, \xi$, where ξ is an arbitrary function of the class $C_c^\infty(\Omega)$, which we take positive in Ω. It follows that

$$\int_\Omega k(\theta_\delta)\, \nabla\theta_\delta \cdot \nabla\xi\, g_\gamma(\theta_\delta) + \int_\Sigma G(\sigma\, \theta_\delta^4)\, \xi\, g_\gamma(\theta_\delta) + R_\delta = \int_\Omega \tilde{f}_\delta\, \xi\, g_\gamma(\theta_\delta)\,,$$

with the notation

$$R_\delta := \int_\Omega k(\theta_\delta)\, |\nabla\theta_\delta|^2\, \xi\, g'_\gamma(\theta_\delta) \leq 0\,.$$

Taking the sign of R_δ into account, we obtain the inequality

$$\int_\Omega k(\theta_\delta)\,\nabla\theta_\delta\cdot\nabla\xi\,g_\gamma(\theta_\delta)+\int_\Sigma G(\sigma\,\theta_\delta^4)\,\xi\,g_\gamma(\theta_\delta)\geq\int_\Omega\tilde f_\delta\,\xi\,g_\gamma(\theta_\delta)\,. \tag{4.42}$$

We pass to the limit $\delta\to 0$ in this last expression. With the help of Lemma 4.4.2 below, we have

$$\begin{aligned}\int_\Sigma G(\sigma\,\theta_\delta^4)\,\xi\,g_\gamma(\theta_\delta)&=\int_\Sigma\epsilon\,\sigma\,\frac{\theta_\delta^4}{1+\gamma\,\theta_\delta^4}\,\xi-\int_\Sigma\epsilon\,\tilde{\mathbf{H}}(\sigma\,\theta_\delta^4)\,\xi\,g_\gamma(\theta_\delta)\\&\longrightarrow\int_\Sigma\epsilon\,\sigma\,\frac{\theta^4}{1+\gamma\,\theta^4}\,\xi-\int_\Sigma\epsilon\,\sigma\,u\,\xi\,g_\gamma(\theta)\,.\end{aligned}$$

We obtain

$$\int_\Omega k(\theta)\,\nabla\theta\cdot\nabla\xi\,g_\gamma(\theta)+\int_\Sigma\epsilon\,\sigma\,\frac{\theta^4}{1+\gamma\,\theta^4}\,\xi-\int_\Sigma\epsilon\,\sigma\,u\,\xi\,g_\gamma(\theta)\geq\int_\Omega\tilde f\,\xi\,g_\gamma(\theta)\,.$$

For all $t\in\mathbb{R}^+$, $g_\gamma(t)\nearrow 1$ as $\gamma\to 0$. By monotone convergence, passage to the limit in the last inequality yields

$$\int_\Omega k(\theta)\,\nabla\theta\cdot\nabla\xi+\int_\Sigma\epsilon\,\sigma\,\theta^4\,\xi-\int_\Sigma\epsilon\,\sigma\,u\,\xi\geq\int_\Omega\tilde f\,\xi\,. \tag{4.43}$$

Comparing the relations (4.41) and (4.43) we find that

$$\int_\Sigma\epsilon\,\sigma\,\theta^4\,\xi\geq\lim_{\delta\to 0}\int_\Sigma\epsilon\,\sigma\,\theta_\delta^4\,\xi\,,$$

for all $\xi\in C_c^\infty(\Omega)$ such that $\xi\geq 0$ in Ω. With the help of Fatou's lemma and of the pointwise convergence (4.40), we have

$$\lim_{\delta\to 0}\int_\Sigma\epsilon\,\sigma\,\theta_\delta^4\xi=\int_\Sigma\epsilon\,\sigma\,\theta^4\,\xi\,. \tag{4.44}$$

Now, in view of (3.49), it is possible to choose $\xi\in C_c^\infty(\Omega)$ such that $\xi\geq 0$ in Ω and $\xi=1$ on Σ. It then follows from (4.44) and Lemma 4.4.3 below that

$$\theta_\delta^4\longrightarrow\theta^4\ \text{in}\ L^1(\Sigma)\,,$$

proving the last assertion and the proposition. □

Lemma 4.4.2. Let a_k, $a\in L^\infty(\Omega)$ such that $\|a_k\|_{L^\infty(\Omega)}\leq A$ for all $k\in\mathbb{N}$. Let b_k, $b\in L^1(\Omega)$. Suppose that

$$a_k\to a\ \text{a. e.}\qquad b_k\rightharpoonup b\ \text{in}\ L^1(\Omega)\,.$$

Then, $a_k\,b_k\rightharpoonup a\,b$ in $L^1(\Omega)$.

Proof. See Giaquinta et al. [1998], I.2.4. Proposition 1. □

Lemma 4.4.3. Let u_k, $u\in L^1(\Omega)$ be such that

$$u_k\to u\ \text{a. e.}\qquad \|u_k\|_{L^1(\Omega)}\to\|u\|_{L^1(\Omega)}\,.$$

Then $u_k\to u$ strongly in $L^1(\Omega)$.

Proof. See Giaquinta et al. [1998], I.2.3 Proposition 4. □

The parabolic case Passage to the limit is essentially more difficult in the parabolic case. As a matter of fact, the integral operator K does not regularize the problem in time, so that compactness properties such as Lemma 4.1.14, (4) cannot be expected. In the paper Druet [2008a], we did not even obtain the uniform bound $\|\theta_\delta\|_{L^4(\mathcal{S})} \leq C$ for the sequence of approximate solution. However, passage to the limit can still be accomplished in a satisfactory sense.

We assume that for all values of a parmeter $\delta > 0$, there exists $\theta_\delta \in W^{1,0}_p(Q)$, $(p \geq 5)$ such that

$$-\int_Q \theta_\delta \frac{\partial \xi}{\partial t} + \int_Q u_\delta \cdot \nabla\theta_\delta\, \xi + \int_Q k(\theta_\delta)\, \nabla\theta_\delta \cdot \nabla\xi + \int_{\mathcal{S}} G(\sigma\, \theta_\delta^4)\, \xi = \int_\Omega \theta_0\, \xi + \int_Q \tilde{f}_\delta\, \xi\,, \tag{4.45}$$

for all $\xi \in W^1_p(Q)$ such that $\xi(T) = 0$. Here again, $\tilde{f}_\delta \in L^2(Q)$ is an approximation of an L^1-force term $\tilde{f}$, and it is known that

$$\tilde{f}_\delta \longrightarrow \tilde{f} \quad \text{in } L^1(Q)\,. \tag{4.46}$$

The vectors $\{u_\delta\} \in [V^{1,0}_2(Q)]^3$ satisfy div $u_\delta = 0$ in Q, $u_\delta \cdot \vec{n} = 0$ on $]0, T[\times\partial\Omega$. In addition, we assume that there exists $u \in [V^{1,0}_2(Q)]^3$ such that

$$u_\delta\, \theta_\delta \rightharpoonup u\, \theta \quad \text{in } [L^1(Q)]^3\,. \tag{4.47}$$

Proposition 4.4.4. Let Ω be an enclosure that satisfies (3.49). Assume that $\theta_\delta \in W^{1,0}_p(Q)$, $(p \geq 5)$ satisfies (4.45), and that $\theta_\delta \geq 0$ almost everywhere in Q.

In addition, let $\theta_\delta \rightharpoonup \theta$ in $W^{1,0}_r(Q)$ for some $1 < r < \infty$, and let $\theta_\delta \longrightarrow \theta$ almost everywhere in Q and on $\mathcal{S}$. If (4.46) and (4.47) are valid, we can prove for a subsequence that

$$\int_{\mathcal{S}} G(\sigma\, \theta_\delta^4)\, \xi \longrightarrow \int_{\mathcal{S}} \sigma\, \theta^4\, G(\xi)\,,$$

for all $\xi \in W^1_{\infty,\mathcal{C}}(Q)$ such that $\xi = 0$ in $\{T\} \times \Omega$, and such that

$$\int_0^T \max_{\overline{\Omega}} |\xi(t)| \int_{\mathcal{S}} \theta^4(t)\, dt\; < \infty\,. \tag{4.48}$$

Proof. For all $\xi \in W^1_{\infty,\mathcal{C}}(Q)$ such that $\xi(T) = 0$, we can pass to the limit $\delta \to 0$ in (4.45). We obtain the existence of $\lim_{\delta\to 0} \int_{\mathcal{S}} G(\sigma\, \theta_\delta^4\,)\, \xi$, as well as the identity

$$-\int_Q \theta \frac{\partial \xi}{\partial t} - \int_Q u\, \theta\, \nabla\xi + \int_Q k(\theta)\, \nabla\theta \cdot \nabla\xi + \lim_{\delta\to 0} \int_{\mathcal{S}} G(\sigma\, \theta_\delta^4\,)\, \xi = \int_\Omega \theta_0\, \xi + \int_Q \tilde{f}\, \xi\,. \tag{4.49}$$

We start again from (4.45). For a while, we consider a fixed δ and write θ, $\tilde{f}$, u instead of θ_δ, $\tilde{f}_\delta$, u_δ.

Employing standard techniques (see Lemma A.3.1 below) for the Steklov averaging operator $(\cdot)_{(h)}$, we can prove the relation

$$\int_{Q_{t_1}} \frac{\partial \theta_{(h)}}{\partial t}\,\xi + \int_{Q_{t_1}} \{u\cdot\nabla\theta\}_{(h)}\,\xi + \int_{Q_{t_1}} \{k(\theta)\,\nabla\theta\}_{(h)}\cdot\nabla\xi + \int_{\mathcal{S}_{t_1}} \{G(\sigma\,\theta^4)\}_{(h)}\,\xi$$
$$= \int_{Q_{t_1}} \{\tilde f\}_{(h)}\,\xi\,, \tag{4.50}$$

for all parameter $h \in]0,T[$, for all $\xi \in W^{1,0}_p(Q)$, and all $t_1 < T-h$. For $\gamma > 0$, consider the function

$$g(s) = g_\gamma(s) := \frac{1}{1+\gamma\,s^4} \quad \text{for } s \in \mathbb{R}^+\,.$$

Taking an arbitrary $\tilde\xi \in W^1_{\infty,\mathcal{C}}(Q)$, such that $\tilde\xi(T) = 0$, and $\tilde\xi \geq 0$ in Q, we may choose in (4.50) the test function $\xi_{\gamma,h} = g_\gamma(\theta_{(h)})\,\tilde\xi$.

Denoting by $F = F_\gamma$ the primitive function of g_γ that vanishes at zero, and not indicating for a while the dependence on the parameters γ, we can write

$$-\int_{Q_{t_1}} F(\theta_{(h)})\,\frac{\partial\tilde\xi}{\partial t} + \int_{Q_{t_1}} \{u\,\theta\}_{(h)}\cdot\nabla(\tilde\xi\,g(\theta_{(h)})) + \int_{Q_{t_1}} \{k(\theta)\,\nabla\theta\}_{(h)}\cdot\nabla\tilde\xi\,g(\theta_{(h)}) + R_h$$
$$+ \int_{\mathcal{S}_{t_1}} \{G(\sigma\,\theta_\delta^4\,)\}_{(h)}\,\tilde\xi\,g(\theta_{(h)}) = \int_\Omega F(\theta_{(h)}(0))\,\tilde\xi(0) + \int_{Q_{t_1}} \{\tilde f\}_{(h)}\,\tilde\xi\,g(\theta_{(h)})\,, \tag{4.51}$$

with the notation

$$R_h := \int_{Q_{t_1}} \{k(\theta)\,\nabla\theta\}_{(h)}\cdot\nabla\theta_{(h)}\,\tilde\xi\,g'(\theta_{(h)})\,.$$

Note that, as $h \to 0$,

$$R_h \longrightarrow \int_{Q_{t_1}} k(\theta)\,|\nabla\theta|^2\,\tilde\xi\,g'(\theta) \leq 0\,,$$

because g is decreasing. Therefore, passing to the limit $h \to 0$ in (4.51) we find the inequality

$$-\int_{Q_{t_1}} F(\theta)\,\frac{\partial\tilde\xi}{\partial t} - \int_{Q_{t_1}} u\,\theta\cdot\nabla(\tilde\xi\,g(\theta)) + \int_{Q_{t_1}} k(\theta)\,\nabla\theta\cdot\nabla\tilde\xi\,g(\theta) + \int_{\mathcal{S}_{t_1}} G(\sigma\,\theta^4)\,\tilde\xi\,g(\theta)$$
$$\geq \int_\Omega F(\theta(0))\,\tilde\xi(0) + \int_{Q_{t_1}} \tilde f\,\tilde\xi\,g(\theta)\,.$$

Observe that

$$\int_{Q_{t_1}} u\,\theta\cdot\nabla(\tilde\xi\,g(\theta)) = \int_{Q_{t_1}} u\,\theta\,g(\theta)\cdot\nabla\tilde\xi + \int_{Q_{t_1}} u\cdot\nabla F(\theta)\,\tilde\xi$$
$$= \int_{Q_{t_1}} u\,(\theta\,g(\theta) - F(\theta))\cdot\nabla\tilde\xi\,,$$

where we used integration by parts and the fact that u is divergence free. It follows that

$$-\int_{Q_{t_1}} F(\theta)\,\frac{\partial\tilde{\xi}}{\partial t} - \int_{Q_{t_1}} u\,(\theta\, g(\theta) - F(\theta))\cdot\nabla\tilde{\xi}\; + \int_{Q_{t_1}} k(\theta)\,\nabla\theta\cdot\nabla\tilde{\xi}\, g(\theta)$$
$$+\int_{\mathbb{S}_{t_1}} G(\sigma\,\theta^4)\,\tilde{\xi}\, g(\theta) \geq \int_{\Omega} F(\theta_0)\,\tilde{\xi}(0) + \int_{Q_{t_1}} \tilde{f}\,\tilde{\xi}\, g(\theta)\,. \tag{4.52}$$

In (4.52), we recall that $\theta = \theta_\delta$, $u = u_\delta$ and $\tilde{f} = \tilde{f}_\delta$. We observe that by the definition of g, we have

$$\int_{\mathbb{S}} G(\sigma\,\theta_\delta^4\,)\,\tilde{\xi}\, g(\theta_\delta) = \int_{\mathbb{S}} \sigma\,\theta_\delta^4\,\tilde{\xi}\, g(\theta_\delta) - \int_{\mathbb{S}} \sigma\,\theta_\delta^4\,\mathbf{H}(\tilde{\xi}\, g(\theta_\delta))$$
$$= \int_{\mathbb{S}} \sigma\,\frac{\theta_\delta^4}{1+\gamma\,\theta_\delta^4}\,\tilde{\xi} - \int_{\mathbb{S}} \sigma\,\theta_\delta^4\,\mathbf{H}(\tilde{\xi}\, g(\theta_\delta))\,.$$

This implies that

$$\lim_{\delta\to 0}\int_{\mathbb{S}} G(\sigma\,\theta_\delta^4)\,\tilde{\xi}\, g(\theta_\delta) \leq \int_{\mathbb{S}} \frac{\sigma\,\theta^4}{1+\gamma\,\theta^4}\,\tilde{\xi} - \liminf_{\delta\to 0}\int_{\mathbb{S}} \sigma\,\theta_\delta^4\,\mathbf{H}(\tilde{\xi}\;g(\theta_\delta))\,, \tag{4.53}$$

where we made use of the dominated convergence theorem.

On the other hand, by Fatou's lemma, and by the fact that $g(\theta_\delta) \to g(\theta)$ in $L^1(\Sigma)$, we have

$$\liminf_{\delta\to 0}\int_{\mathbb{S}} \sigma\,\theta_\delta^4\,\mathbf{H}(\tilde{\xi}\, g(\theta_\delta)) \geq \int_{\mathbb{S}} \sigma\;\liminf_{\delta\to 0}\Big\{\,\theta_\delta^4\,\mathbf{H}(\tilde{\xi}\, g(\theta_\delta))\Big\}$$
$$\geq \int_{\mathbb{S}} \sigma\,\theta^4\;\liminf_{\delta\to 0}\mathbf{H}(\tilde{\xi}\, g(\theta_\delta)) \geq \int_{\mathbb{S}} \sigma\,\theta^4\,\mathbf{H}(\tilde{\xi}\, g(\theta))\,. \tag{4.54}$$

Returning in (4.53), we can write

$$\lim_{\delta\to 0}\int_{\mathbb{S}} G(\sigma\,\theta_\delta^4)\,\tilde{\xi}\, g(\theta_\delta) \leq \int_{\mathbb{S}} \sigma\,\theta^4\, g(\theta)\,\tilde{\xi} - \int_{\mathbb{S}} \sigma\,\theta^4\,\mathbf{H}(\tilde{\xi}\, g(\theta))\,.$$

The relation (4.52) is therefore preserved in the limit $\delta \to 0$, and we can write

$$-\int_Q F(\theta)\,\frac{\partial\tilde{\xi}}{\partial t} - \int_Q u\cdot\nabla\tilde{\xi}\,(\theta\, g(\theta) - F(\theta)) + \int_Q k(\theta)\,\nabla\theta\cdot\nabla\tilde{\xi}\, g(\theta) + \int_{\mathbb{S}} \sigma\,\theta^4\, g(\theta)\,\tilde{\xi}$$
$$\geq \int_\Omega F(\theta_0)\,\tilde{\xi}(0) + \int_Q \tilde{\xi}\, g(\theta) + \int_{\mathbb{S}} \sigma\,\theta^4\, H(\tilde{\xi}\, g(\theta))\,. \tag{4.55}$$

We now recall that $F = F_\gamma$ and $g = g_\gamma$. By the definition of these functions, we see easily that F_γ monotonely increases to the identity, and that g_γ monotonely increases to the constant function 1. This last property ensures that

$$\int_{\mathbb{S}} \sigma\,\theta^4\, g_\gamma(\theta)\,\tilde{\xi} \longrightarrow \int_{\mathbb{S}} \sigma\,\theta^4\,\tilde{\xi}\,,$$

for $\gamma \to 0$. As in (4.54), we can conclude that

$$\liminf_{\gamma\to 0} \int_{\mathcal{S}} \sigma\, \theta^4\, \mathbf{H}(\tilde{\xi}\, g_\gamma(\theta)) \geq \int_{\mathcal{S}} \sigma\theta^4\, \mathbf{H}(\tilde{\xi})\,.$$

In the limit $\gamma \to 0$ of (4.55), we thus obtain the inequality

$$\begin{aligned} -\int_Q \theta\, \frac{\partial \tilde{\xi}}{\partial t} - \int_Q u\cdot \nabla\tilde{\xi}\,\theta + \int_Q k(\theta)\,\nabla\theta\cdot\nabla\tilde{\xi} + \int_{\mathcal{S}} \sigma\,\theta^4\,\tilde{\xi} \\ \geq \int_\Omega \theta_0\,\tilde{\xi}(0) + \int_Q \tilde{f}\,\tilde{\xi} + \int_{\mathcal{S}} \sigma\,\theta^4\, H(\tilde{\xi})\,. \end{aligned} \tag{4.56}$$

Now, considering a test function $\tilde{\xi}$ with the additional property (4.48), we see that the integral $\int_{\mathcal{S}} \sigma\,\theta^4\,\tilde{\xi}$ is finite. It follows from (4.56) that $\int_{\mathcal{S}} \sigma\,\theta^4\, H(\tilde{\xi})$ has also to be finite. Thus, for this $\tilde{\xi}$, we have

$$-\int_Q \theta\, \frac{\partial \tilde{\xi}}{\partial t} - \int_Q u\cdot \nabla\tilde{\xi}\,\theta + \int_Q k(\theta)\,\nabla\theta\cdot\nabla\tilde{\xi} + \int_{\mathcal{S}} \sigma\,\theta^4\, G(\tilde{\xi}) \geq \int_\Omega \theta_0\,\tilde{\xi}(0) + \int_Q \tilde{f}\,\tilde{\xi}\,. \tag{4.57}$$

Comparing (4.49) and (4.57), where we choose $\xi = \tilde{\xi}$, we get

$$\lim_{\delta\to 0} \int_{\mathcal{S}} \sigma\,\theta_\delta^4\, G(\xi) = \lim_{\delta\to 0} \int_{\mathcal{S}} G(\sigma\,\theta_\delta^4)\,\xi \leq \int_{\mathcal{S}} \sigma\,\theta^4\, G(\xi)\,, \tag{4.58}$$

for all $\xi \in W^1_{\infty,\mathcal{C}}(Q)$ such that $\xi(T) = 0$, such that $\xi \geq 0$ in Q, and such that (4.48) is satisfied. Elementarily, we therefore have that

$$\lim_{\delta\to 0} \int_{\mathcal{S}} \sigma\,\theta_\delta^4\, G(\xi) \geq \int_{\mathcal{S}} \sigma\,\theta^4\, G(\xi)\,, \tag{4.59}$$

for all $\xi \in W^1_{\infty,\mathcal{C}}(Q)$ with (4.48) such that $\xi(T) = 0$ and $\xi \leq 0$ in Q.

Given $\xi \in W^1_{\infty,\mathcal{C}}(Q)$ with (4.48) such that $\xi(T) = 0$ and $\xi \geq 0$ in Q, we use the construction of Lemma 4.4.5 below to find the function $\bar{\xi}$. We verify that

$$\int_0^T \max_{\overline{\Omega}} |\bar{\xi}(t)| \int_\Sigma \theta^4(t)\,dt \leq 2 \int_0^T \max_{\overline{\Omega}} |\xi(t)| \int_\Sigma \theta^4(t)\,dt < \infty\,,$$

so that $\bar{\xi}$ also satisfies (4.48).

Using (4.58) and (4.59) we can write, for all $\xi \in W^1_{\infty,\mathcal{C}}(Q)$ with (4.48) such that $\xi(T) = 0$ and $\xi \geq 0$ in Q, that

$$\int_{\mathcal{S}} \sigma\,\theta^4\, G(\xi) = \int_{\mathcal{S}} \sigma\,\theta^4\, G(\bar{\xi}) \leq \lim_{\delta\to 0} \int_{\mathcal{S}} \sigma\,\theta_\delta^4\, G(\bar{\xi}) = \lim_{\delta\to 0} \int_{\mathcal{S}} \sigma\,\theta_\delta^4\, G(\xi) \leq \int_{\mathcal{S}} \sigma\,\theta^4\, G(\xi)\,,$$

which shows us that

$$\lim_{\delta\to 0} \int_{\mathcal{S}} \sigma\,\theta_\delta^4\, G(\xi) = \int_{\mathcal{S}} \sigma\,\theta^4\, G(\xi)\,, \tag{4.60}$$

for all $\xi \in W^1_{\infty,\mathcal{C}}(Q)$ with (4.48) such that $\xi(T) = 0$. $\square$

Lemma 4.4.5. Assume that (3.1) is valid. Assume that Σ belongs to $\mathcal{C}^1$ piecewise and satisfies (3.49).

Then for an arbitrary ψ in $W^1_{\infty,\mathcal{C}}(Q)$ such that $\psi \geq 0$ in Q and $\psi = 0$ in $\{T\} \times \Omega$, there exists

$$\bar{\psi} \in W^1_{\infty,\mathcal{C}}(Q)\,, \begin{cases} \bar{\psi} \leq 0 & \text{in } Q\,, \\ \bar{\psi} = 0 & \text{in } \{T\} \times \Omega\,, \end{cases}$$

such that $G(\psi) = G(\bar{\psi})$ on $\mathcal{S}$. In addition, $\|\bar{\psi}\|_{L^\infty(Q)} \leq \|\psi\|_{L^\infty(Q)}$.

Proof. Consider an arbitrary ψ in $W^1_{\infty,\mathcal{C}}(Q)$ such that $\psi \geq 0$ in Q, $\psi = 0$ in $\{T\} \times \Omega$. Define

$$\zeta(t) := \max_{y \in \overline{\Omega}} \psi(t, y)\,.$$

It is then elementarily verified that ζ is a Lipschitz continuous function on $[0,\, T]$, that $\zeta(T) = 0$, and that

$$\psi(t,\, x) - \zeta(t) \leq 0 \text{ in } Q\,.$$

We now choose a smooth function $\phi \in C^\infty(\overline{\Omega})$, positive in Ω, such that

$$\phi \equiv 1 \text{ on } \Sigma, \qquad \phi \equiv 0 \text{ on } \Gamma, \qquad 0 \leq \phi \leq 1 \text{ in } \Omega\,.$$

Let $\bar{\psi}(t,x) := (\psi(t,x) - \zeta(t))\,\phi(x)$. Then, $\bar{\psi}$ is negative in Q, vanishes on $\mathcal{C}$ and in $\{T\} \times \Omega$. In addition, $\bar{\psi} \in W^1_\infty(Q)$. Since Ω satisfies the assumption (3.1), observe that $G(1) \equiv 0$ by Lemma 4.1.8, (4). Thus, we can write for all $t \in [0,\, T]$ that

$$G(\bar{\psi}(t)) = G((\psi(t) - \zeta(t))\,\phi) = G(\psi(t)) - \zeta(t)\,G(1) = G(\psi(t)) \text{ on } \Sigma\,.$$

Finally,

$$\bar{\psi}(t,\, x) \geq -\phi(x)\,\zeta(t) \geq -\|\psi\|_{L^\infty(Q)} \text{ in } Q\,.$$

$\square$

Chapter 5

Auxiliary results II. Higher integrability of the Lorentz force $j \times B$

he results of this chapter have been published in Druet [2007a]. They do not claim to be essentially new, but rather want to provide a useful survey of the matter.

The purpose of this chapter is to formulate some consequences of recent regularity results that are relevant for the mathematical theory of the MHD equations. More specifically, we want to show how the theory of the papers Zanger [2000], Elschner et al. [2007] can help us to deal with the difficulties that arise from coupling the weak formulation of Maxwell's system to the equation of momentum balance in complex geometries.

For the field H that solves in the weak sense the stationary Maxwell system with linear constitutive relations (see the paragraph (3.1.1), the generalized theory of electromagnetics gives the following basic informations (see for example Duvaut and Lions [1976], Picard and Milani [1999] or Bossavit [2004]):

$$H \in V_{\mu,0}(\tilde{\Omega}) := \left\{ \psi \in [L^2(\tilde{\Omega})]^3 \;\middle|\; \operatorname{curl} \psi \in [L^2(\tilde{\Omega})]^3,\ \operatorname{div}(\mu\,\psi) = 0,\ \gamma_n(\mu\,\psi) = 0 \text{ on } \partial\tilde{\Omega} \right\}, \tag{5.1}$$

where the operators *curl*, *div* are intended in the generalized sense, and γ_n denotes the trace in normal direction.

For the study of the coupled problem (P_{st}), the functional setting (5.1) (cp. the paragraph 3.2.1) can lead to considerable difficulties. For the electromagnetic force in (3.2), we have in view of (3.6) and (3.9)

$$j \times B = \operatorname{curl} H \times \mu\, H\,.$$

Therefore, if our knowledge about the regularity of H is limited to (5.1), we cannot in general expect more than $j \times B \in [L^1(\tilde{\Omega})]^3$.

Most papers about magnetohydrodynamics try to avoid this difficulty, which is possible with the help of reinforced assumptions on the regularity of the function μ, and on the structure of the domain $\tilde{\Omega}$. A *first idea*, applied e. g. in Duvaut and Lions [1972], Sermange and Temam [1983], consists in supposing that μ is a *globally smooth* function in the domain $\tilde{\Omega}$. From (5.1) it then follows that

$$\operatorname{div} H = \frac{-\nabla\mu}{\mu} \cdot H \in L^2(\tilde{\Omega})\,.$$

Whenever the function μ satisfies (3.44), a vector field H that satisfies (5.1) then belongs to the space

$$V := \left\{\psi \in [L^2(\tilde{\Omega})]^3 \,\middle|\, \operatorname{curl}\psi \in [L^2(\tilde{\Omega})]^3,\ \operatorname{div}\psi \in L^2(\tilde{\Omega}),\ \gamma_n(\psi) = 0 \text{ on } \partial\tilde{\Omega}\right\}. \tag{5.2}$$

If the boundary $\partial\tilde{\Omega}$ is of class $\mathcal{C}^2$, then the topological identity

$$V = \left\{H \in [H^1(\tilde{\Omega})]^3 \,\middle|\, H \cdot \vec{n} = 0 \text{ almost everywhere on } \partial\tilde{\Omega}\right\}, \tag{5.3}$$

is valid (a proof is given in Duvaut and Lions [1976], chapter 7, Th. 6.1). With the help of Sobolev's embedding theorems, we obtain in this case that $H \in [L^6(\tilde{\Omega})]^3$. This gives that $\operatorname{curl} H \times B \in [L^{3/2}(\tilde{\Omega})]^3$. Thanks to recent advances in regularity theory (see e. g. the paper Amrouche et al. [1998] and the references therein), this type of result can be extended to less regular, Lipschitz domains. Still assuming that the permeability μ is globally smooth, one proves for example that

$$V \hookrightarrow [H^{1/2}(\tilde{\Omega})]^3\,, \tag{5.4}$$

with continuous embedding (see Monk [2003], Theorem 3.47, for a proof). This gives for a vector field H that satisfies (5.1) that $H \in [L^3(\tilde{\Omega})]^3$. One obtains that $\operatorname{curl} H \times B$ belongs to $[L^{6/5}(\tilde{\Omega})]^3$ even in Lipschitz domains, which is still sufficient for solving the Navier-Stokes equations via standard theory.

The smoothness of the permeability μ cannot always be assumed. In real-life applications, the magnetic permeability jumps at interfaces that separate different materials. It is necessary to take into account the transmission conditions (3.18) at the interfaces.

The authors of the paper Ladyzhenskaja and Solonnikov [1960] considered a setting with two disjoint subdomains $\tilde{\Omega} = \overline{\Omega_1} \cup \Omega_2$, where the set Ω_1 is supposed to be simply connected and compactly included in $\tilde{\Omega}$. Under the assumption that the outer boundary $\partial\tilde{\Omega}$, as well as the *interface* $\partial\Omega_1$ are of class $\mathcal{C}^2$, one can prove the topological identity

$$V_{\mu,0}(\tilde{\Omega}) = V_{\mu,0}(\tilde{\Omega}) \cap \bigcap_{i=1}^{2}[W^{1,2}(\Omega_i)]^3\,. \tag{5.5}$$

This result was confirmed by other methods in the more recent papers Meir and Schmidt [1996], Meir and Schmidt [1999]. One must note, however, that restriction to interfaces that are globally in the class $\mathcal{C}^2$ (or at least in $\mathcal{C}^{1,1}$) excludes most situation that one expects to find in complex applications, such as triple jump points of the magnetic permeability, or interfaces with corners (cp. the picture 3.3).

In the present chapter, our aim is to take advantage of recent regularity results to derive weaker conditions than in Ladyzhenskaja and Solonnikov [1960] on the data pair $(\mu,\, \tilde{\Omega})$ under which we can obtain the higher integrability of $j \times B$.

We present two different sets of hypotheses that yield the existence of a number $q > 3$ such that the space $V_{\mu,0}(\tilde{\Omega})$ embeds continuously into $[L^q(\tilde{\Omega})]^3$. This gives that $curl\, H \times B \in [L^r(\tilde{\Omega})]^3$ for some $r > 6/5$.

First, exploiting the regularity theory of Elschner et al. [2007], the higher integrability follows if we require (3.50).

Second, exploiting the results of Zanger [2000], the higher integrability follows without further conditions on the interfaces, provided that the domain $\tilde{\Omega}$ is Lipschitzian and that the function $\mu \in L^\infty(\tilde{\Omega})$ is nearly constant (in a sense to be made more precise below, cp. (3.51)). The latter situation quite often occurs in practice, since the ratio of the magnetic permeability of nonmagnetic materials to the magnetic permeability of the vacuum is nearly one.

5.1 Embedding results for vector fields that satisfy a curl and a div constraint

Several embedding results have been stated in the past for vector fields that satisfy a *curl* and a *div* constraint. In general a constraint on the normal or on the tangential values at the boundary is also needed. A typical example is given by the embedding (5.3) of the space V defined in (5.2), which relies on the inequality

$$\|\nabla\psi\|_{[L^2(U)]^9} \leq c\,(\|\operatorname{curl}\psi\|_{[L^2(U)]^3} + \|\operatorname{div}\psi\|_{L^2(U)})\,, \tag{5.6}$$

valid whenever the domain $U \subset \mathbb{R}^3$ is of class $\mathcal{C}^2$ (see Duvaut and Lions [1976], Ch. 7, Th. 6.1 for a proof). The inequality (5.6) is known in the context of differential geometry as Gaffney's inequality, see Picard [1984]). Inequalities of this type can be generalized in smooth domains to the case $1 < p < +\infty$, as was shown in von Wahl [1992], Th. 2.1, and to Sobolev spaces of fractional order.

In the case nonsmooth domains, these results mostly extend to convex polyhedra (see Girault and Raviart [1986] and references), but examples of Lipschitz domains in three space dimensions are known for which (5.6) fails: these are basically domains with re-entrant corners on which the Dirichlet problem for the Laplace operator do not have a solution in H^2. One can still hope, though, to prove an embedding result in higher L^p-spaces, i. e. an inequality of the type

$$\|\psi\|_{[L^q(U)]^9} \leq c\,(\|\operatorname{curl}\psi\|_{[L^p(U)]^3} + \|\operatorname{div}\psi\|_{L^p(U)})\,,$$

with $q > p$. An example of a similar result obtained via embedding results for Sobolev spaces of fractional order is given by (5.4). In the following of this preliminary section, we first recall basic notions concerning the generalized operators *curl* and *div*, and then investigate embedding results that can be obtained directly.

The generalized operators *curl* and *div*.

We at first recall the definitions of the generalized differential operators *curl* and *div*.

Definition 5.1.1. Let $U \subset \mathbb{R}^3$ be a bounded domain, and $1 \leq p \leq \infty$.

(1) For a vector field $\psi \in [L^p(U)]^3$, we write curl $\psi \in [L^p(U)]^3$ if there exists a $\xi \in [L^p(U)]^3$ such that

$$\int_U \psi \cdot \operatorname{curl} \phi = \int_U \xi \cdot \phi \,,$$

for all $\phi \in [C_c^\infty(U)]^3$. The uniquely determined vector field ξ is called the generalized curl of ψ, and we define curl $\psi := \xi$.

(2) For a vector field $\psi \in [L^p(U)]^3$, we write div $\psi \in L^p(U)$ if there exists a function $\zeta \in L^p(U)$ such that

$$\int_U \psi \cdot \nabla\phi = -\int_U \zeta\,\phi \,,$$

for all $\phi \in C_c^\infty(U)$. The uniquely determined function ζ is called the generalized divergence of ψ, and we define div $\psi := \zeta$.

For a bounded domain $U \subset \mathbb{R}^3$, we then introduce

$$L^p_{\mathrm{curl}}(U) := \left\{ \psi \in [L^p(U)]^3 \,\middle|\, \operatorname{curl}\psi \in [L^p(U)]^3 \right\},$$

$$L^p_{\mathrm{div}}(U) := \left\{ \psi \in [L^p(U)]^3 \,\middle|\, \operatorname{div}\psi \in L^p(U) \right\},$$

where the operators *curl* and *div* are intended in the sense of Definition 5.1.1. These spaces are Banach spaces with respect to the graph topologies

$$\|\psi\|_{L^p_{\mathrm{curl}}(U)} := \|\psi\|_{[L^p(U)]^3} + \|\operatorname{curl}\psi\|_{[L^p(U)]^3} \,,$$

$$\|\psi\|_{L^p_{\mathrm{div}}(U)} := \|\psi\|_{[L^p(U)]^3} + \|\operatorname{div}\psi\|_{L^p(U)} \,. \tag{5.7}$$

For $p = 2$, they are Hilbert spaces.

For vector fields that belong to a space (5.7), it is possible to define trace operators. Denoting by $\vec{n}$ the outward-pointing unit normal to ∂U, we have for ϕ, $\psi \in [C^\infty(\overline{U})]^3$ the well-known formula

$$\int_U \psi \cdot \operatorname{curl}\phi - \int_U \operatorname{curl}\psi \cdot \phi = -\int_{\partial U} (\psi \times \vec{n}) \cdot \phi =: -\big\langle \gamma_\tau(\psi),\, \phi \big\rangle \,.$$

Thanks to results for the density of the smooth functions in the spaces (5.7), it can be shown (see for example Duvaut and Lions [1976], Picard and Milani [1999]) that the operator γ_τ extends to a linear bounded operator on the space $L^2_{\mathrm{curl}}(U)$. For $\psi \in L^2_{\mathrm{curl}}(U)$, we then call $\gamma_\tau(\psi)$ the *trace* of ψ. In general, this trace does not need to be identical to an integrable function on the boundary. Nevertheless, for ϕ, $\psi \in L^2_{\mathrm{curl}}(U)$, we often abuse notation and write $\int_{\partial U} (\psi \times \vec{n}) \cdot \phi$ instead of $\big\langle \gamma_\tau(\psi),\, \phi \big\rangle$.

Similarly, for $\psi \in [C^\infty(\overline{U})]^3$ and $\phi \in C^\infty(\overline{U})$, we have the formula

$$\int_U \psi \cdot \nabla\phi + \int_U \operatorname{div}\psi \cdot \phi = \int_{\partial U} \psi \cdot \vec{n}\,\phi =: \big\langle \gamma_n(\psi),\, \phi \big\rangle \,.$$

The operator γ_n extends to a linear bounded operator on the space $L^2_{\mathrm{div}}(U)$. For $\psi \in L^2_{\mathrm{div}}(U)$, we call $\gamma_n(\psi)$ the *trace* of ψ. For $\psi \in L^2_{\mathrm{div}}(U)$ and $\phi \in L^2_{\mathrm{div}}(U)$, we often write $\int_{\partial U} \psi \cdot \vec{n}\,\phi$ instead of $\big\langle \gamma_n(\psi),\, \phi \big\rangle$.

Embedding of $L^p_{\mathrm{curl}}(U) \cap L^p_{\mathrm{div}}(U)$ into $[L^q(U)]^3$ for $(q > p)$.

Let $U \subset \mathbb{R}^3$ be a simply connected bounded domain. For $1 < p,\, \alpha < \infty$, we consider the spaces

$$\mathcal{W}^{p,\alpha}(U) := \left\{ u \in L^p_{\mathrm{curl}}(U) \cap L^p_{\mathrm{div}}(U) \,\middle|\, \gamma_n(u) \in L^\alpha(\partial U) \right\}. \tag{5.8}$$

We denote by p^* the Sobolev embedding exponent

$$p^* := \begin{cases} \frac{3p}{3-p} & \text{if } 1 \le p < 3\,, \\ 1 \le s < \infty \text{ arbitrary} & \text{if } p = 3\,, \\ \infty & \text{if } p > 3\,. \end{cases}$$

In this section, we prove the following main result:

Proposition 5.1.2. Let $U \subset \mathbb{R}^3$ be a simply connected bounded Lipschitz domain. Then, there exists a $q_1 > 3$ such that for all $p \in]q_1',\, q_1[$, the space $\mathcal{W}^{p,\alpha}(U)$ embeds continuously in $[L^\xi(U)]^3$ for $\xi := \min\left\{\frac{3\alpha}{2},\, p^*,\, q_1\right\}$. If the domain U is of class $\mathcal{C}^1$, one may take $q_1 = +\infty$.

In order to prove Proposition 5.1.2, we at first need an extension result.

Lemma 5.1.3. Let $U \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Let $f \in L^p_{\mathrm{div}}(U)$ such that $\operatorname{div} f = 0$ in the sense of the generalized *div* operator. Let $\tilde{U} \subset \mathbb{R}^3$ be a smooth bounded domain such that $U \subset\subset \tilde{U}$.

Then, there exists a $q_1 > 3$ such that for all $p \in]q_1',\, q_1[$, we can find an extension $\tilde{f}$ of f to $\tilde{U} \setminus U$ such that $\tilde{f} \in L^p_{\mathrm{div}}(\tilde{U})$ and

$$\operatorname{div} \tilde{f} = 0 \text{ in the weak sense in } \tilde{U}\,, \quad \tilde{f} \cdot \vec{n} = 0 \text{ on } \partial\tilde{U}\,.$$

In addition, there exists a constant $c = c(U,\, p)$ such that

$$\|\tilde{f}\|_{[L^p(\tilde{U})]^3} \le c\, \|f\|_{[L^p(U)]^3}\,.$$

Proof. Denoting by p' the conjugated exponent to p, we define a functional $F \in [W^{1,p'}(\tilde{U} \setminus U)]^*$ by

$$F(\phi) := \int_{\partial U} f \cdot \vec{n}\, \phi\,.$$

for $\phi \in W^{1,p'}(\tilde{U} \setminus U)$. Since f is weakly divergence-free in U, we see that $F(1) = 0$, and we can find a positive constant c such that

$$\|F\|_{[W^{1,p'}(\tilde{U} \setminus U)]^*} \le c\, \|f\|_{L^p_{\mathrm{div}}(U)} = c\, \|f\|_{[L^p(U)]^3}\,.$$

Theorem 1.6 in Zanger [2000] (see also Proposition 5.3.5 below) provides the existence of some $q_1 > 3$ such that for all $p \in]q_1',\, q_1[$, the Neumann problem to find some $a \in W^{1,p}(\tilde{U} \setminus U)$ such that the relation

$$\int_{\tilde{U} \setminus U} \nabla a \cdot \nabla \phi = F(\phi)\,,$$

holds for all $\phi \in W^{1,p}(\tilde{U} \setminus U)$ possesses a weak solution, which is unique up to constants. In addition, the estimate

$$\|a\|_{W^{1,p}(\tilde{U}\setminus U)} \leq C \, \|F\|_{[W^{1,p'}(\tilde{U}\setminus U)]^*} .$$

is valid, with a constant C that only depends on the Lipschitz constant of the domain $\tilde{U} \setminus U$. We define

$$\tilde{f} := \begin{cases} f & \text{in } U \,, \\ \nabla a & \text{in } \tilde{U} \setminus U \,. \end{cases}$$

It is then easy to verify that this extension has the required properties. □

Proof of Proposition 5.1.2. We consider an arbitrary $u \in \mathcal{W}^{p,\alpha}(U)$.

Define $f := \operatorname{curl} u$. Since $\operatorname{div} f = 0$ almost everywhere in U, we see immediately that $f \in L^p_{\operatorname{div}}(U)$ for all $1 \leq p \leq \infty$. We now choose some p in the range $]q_1', q_1[$, where q_1 is given by Lemma 5.1.3, and we fix some smoothly bounded domain $\tilde{U} \subset \mathbb{R}^3$ such that $U \subset\subset \tilde{U}$. Applying Lemma 5.1.3, we find an extension $\tilde{f} \in L^p_{\operatorname{div}}(\tilde{U})$ such that

$$\tilde{f} = f \text{ in } [L^p(U)]^3, \quad \operatorname{div} \tilde{f} = 0 \text{ weakly in } \tilde{U}, \quad \tilde{f} \cdot \vec{n} = 0 \text{ in the sense of traces on } \partial\tilde{U} \,.$$

In view of Lemma 5.1.3, we have the estimate

$$\|\tilde{f}\|_{[L^p(\tilde{U})]^3} \leq c \, \|f\|_{[L^p(U)]^3} . \tag{5.9}$$

Since the domain $\tilde{U}$ is regular, we can apply Theorem 3.3 in Griessinger [1990], valid for $\mathcal{C}^1$ domains, to find a vector field $A \in [W_0^{1,p}(\tilde{U})]^3$ such that

$$\operatorname{curl} A = \tilde{f} \text{ in } \tilde{U}, \quad \|A\|_{[W_0^{1,p}(\tilde{U})]^3} \leq \bar{c} \, \|\tilde{f}\|_{[L^p(\tilde{U})]^3} ,$$

with a constant $\bar{c}$ that depends on $\tilde{U}$ and on p. Using (5.9), it follows that

$$\|A\|_{[W_0^{1,p}(\tilde{U})]^3} \leq \bar{c} \, \|\operatorname{curl} u\|_{[L^p(U)]^3} . \tag{5.10}$$

Observe that $\operatorname{curl} u = f = \operatorname{curl} A$ almost everywhere in U. Since we assume that U is simply connected, we find a function $r \in W^{1,p}(U)$ such that $u - A = \nabla r$. Our goal is now to obtain an estimate on r.

We define $g := -\operatorname{div}(u - A)$. Since $u \in L^p_{\operatorname{div}}(U)$, we have $g \in L^p(U)$. The function r satisfies

$$\int_U \nabla r \cdot \nabla \phi = F(\phi) \,, \tag{5.11}$$

for all $\phi \in W^{1,p'}(U)$, where F is the functional

$$F(\phi) := \int_U g \, \phi + \int_{\partial U} (u - A) \cdot \vec{n} \, \phi \,.$$

Using Gausses formula, we see that $F(1) = 0$. On the other hand,

$$\Big|F(\phi)\Big| \le \|g\|_{L^p(U)}\,\|\phi\|_{L^{p'}(U)} + \|A\cdot\vec{n}\|_{L^{\frac{2p}{3-p}}(\partial U)}\,\|\phi\|_{L^{\frac{2p'}{3}}(\partial U)} + \|u\cdot\vec{n}\|_{L^\alpha(\partial U)}\,\|\phi\|_{L^{\alpha'}(\partial U)}\,. \tag{5.12}$$

For α and p fixed, we now consider a real number $3 > q > 1$ such that

$$\frac{2\,q}{3-q} \ge \max\left\{\alpha',\, \frac{2\,p'}{3}\right\}. \tag{5.13}$$

This choice of q ensures, on the one hand, the continuity of the embedding

$$W^{1,q}(U) \hookrightarrow L^{\alpha'}(\partial U), \quad W^{1,q}(U) \hookrightarrow L^{\frac{2p'}{3}}(\partial U)\,.$$

On the other hand, we see that for this choice also $\frac{3\,q}{3-q} > p'$, so that the embedding $W^{1,q}(U) \hookrightarrow L^{p'}(U)$ is continuous. From (5.12), we then deduce that

$$\Big|F(\phi)\Big| \le c\,(\|g\|_{L^p(U)} + \|A\|_{[W^{1,p}(U)]^3} + \|u\cdot\vec{n}\|_{L^\alpha(\partial U)})\,\|\phi\|_{W^{1,q}(U)}\,.$$

With the help of (5.10), it now follows that

$$\Big|F(\phi)\Big| \le c\,(\|\operatorname{div} u\|_{L^p(U)} + \|\operatorname{curl} u\|_{[L^p(U)]^3} + \|u\cdot\vec{n}\|_{L^\alpha(\partial U)})\,\|\phi\|_{W^{1,q}(U)}\,,$$

Applying Proposition 5.3.5 (see the appendix), we find the existence of some $q_1 > 3$ such that for all $q \in]q_1',\, q_1[$, the solution r of (5.11) belongs to $W^{1,q'}(U)$. In addition, the estimate

$$\|r\|_{W^{1,q'}(U)} \le c\,\|F\|_{[W^{1,q}(U)]^*} \le C\,(\|\operatorname{div} u\|_{L^p(U)} + \|\operatorname{curl} u\|_{[L^p(U)]^3} + \|u\cdot\vec{n}\|_{L^\alpha(\partial U)})\,,$$

is valid. Note that q_1 is the same as in Lemma 5.1.3. Setting $\xi := \min\{p^*,\, q'\}$, we have

$$\begin{aligned}\|u\|_{[L^\xi(U)]^3} &\le c\,(\|A\|_{[L^{p^*}(U)]^3} + \|\nabla r\|_{[L^{q'}(U)]^3})\\ &\le \bar{C}\,(\|\operatorname{div} u\|_{L^p(U)} + \|\operatorname{curl} u\|_{[L^p(U)]^3} + \|u\cdot\vec{n}\|_{L^\alpha(\partial U)})\,.\end{aligned}$$

It remains to compute the optimal exponent $q' \le q_1$ by taking into account the condition (5.13). We obtain that $q' := \min\left\{\frac{3\alpha}{2},\, p^*,\, q_1\right\}$, and the claim follows. If the domain U is of class $\mathcal{C}^1$, we can apply Theorem 3.3 of Griessinger [1990] directly in U, and we do not need Lemma 5.1.3. By the results of Simader and Sohr [1992], Theorem 1.4, the solution r of (5.11) belongs to $W^{1,q'}(U)$ for $1 < q' < \infty$. Therefore, $q_1 = +\infty$. □

5.2 Conditions for the higher integrability

We now want to study functional spaces more specifically needed for the analysis of the problem (P). In this section, we assume that the domain $\tilde{\Omega}$ is the domain described at the beginning of the section 3.1. Denoting by μ the magnetic permeability in $\tilde{\Omega}$, we assume through the remainder of the paper that μ is a measurable function such that (3.44) is satisfied. We consider the spaces

$$V_\mu(\tilde{\Omega}) := \left\{ \psi \in [L^2(\tilde{\Omega})]^3 \,\middle|\, \operatorname{curl}\psi \in [L^2(\tilde{\Omega})]^3,\ \operatorname{div}(\mu\,\psi) \in L^2(\tilde{\Omega}),\ \gamma_n(\mu\,\psi) = 0 \text{ on } \partial\tilde{\Omega} \right\}, \tag{5.14}$$

$$V_{\mu,0}(\tilde{\Omega}) := \left\{ \psi \in V_\mu(\tilde{\Omega}) \,\middle|\, \operatorname{div}(\mu\,\psi) = 0 \right\}. \tag{5.15}$$

We endow $V_\mu(\tilde{\Omega})$ with the norm of the graph

$$\|\psi\|_{V_\mu(\tilde{\Omega})} := \|\psi\|_{[L^2(\tilde{\Omega})]^3} + \|\operatorname{curl}\psi\|_{[L^2(\tilde{\Omega})]^3} + \|\operatorname{div}(\mu\,\psi)\|_{L^2(\tilde{\Omega})}\,.$$

Obviously, $V_\mu(\tilde{\Omega})$ is a Hilbert space in this topology.

In the introduction, we have emphasized the importance of additional hypotheses on the pair $(\mu,\,\tilde{\Omega})$ for embedding results concerning space of the type of $V_\mu(\tilde{\Omega})$. In this respect, an important class of domains consists of the domains $\overline{\tilde{\Omega}} = \bigcup_{i=0}^m \overline{\tilde{\Omega}_i}$ having the property

$$\begin{cases} \text{If } \tilde{\Omega}_i,\, \tilde{\Omega}_j \subset \tilde{\Omega}_c, \text{ for some } i,\, j \in \{0,\dots,m\},\, i \neq j, \text{ then } \operatorname{dist}(\tilde{\Omega}_i,\, \tilde{\Omega}_j) > 0\,. \\ \text{If } \tilde{\Omega}_i \subset \tilde{\Omega}_c, \text{ for some } i \in \{0,\dots,m\}, \text{ then } \operatorname{dist}(\tilde{\Omega}_i,\, \partial\tilde{\Omega}) > 0\,. \end{cases} \tag{5.16}$$

For the pair $(\mu,\,\tilde{\Omega})$ we want to discuss the following regularity assumptions

$$\begin{cases} \mu_{|_{\tilde{\Omega}_i}} \in \mathcal{C}^1\left(\overline{\tilde{\Omega}_i}\right) & \text{for } i = 0,\dots,m, \\ \partial\tilde{\Omega}_i,\, \partial\tilde{\Omega} \in \mathcal{C}^2 & \text{for } i = 0,\dots,m. \end{cases} \tag{5.17}$$

$$\begin{cases} \mu_{|_{\tilde{\Omega}_i}} \in \mathcal{C}\left(\overline{\tilde{\Omega}_i}\right) & \text{for } i = 0,\dots,m, \\ \partial\tilde{\Omega}_i \setminus \partial\tilde{\Omega} \in \mathcal{C}^1 & \text{for } i = 0,\dots,m, \\ \partial\tilde{\Omega} \in \mathcal{C}^{0,1} & . \end{cases} \tag{5.18}$$

The main result of this section is an embedding result for the space $V_\mu(\tilde{\Omega})$. In order to complete its proof, we first need two auxiliary statements. For a real number $q \in]1,\infty[$, we recall that q' denotes the conjugated exponent to q.

Lemma 5.2.1. Let $\tilde{\Omega}$ be a simply connected Lipschitz domain. Let $\psi \in [L^2(\tilde{\Omega})]^3$ be given, and assume that $p \in W^{1,2}(\tilde{\Omega})$ satisfies for all $\phi \in W^{1,2}(\tilde{\Omega})$ the integral relation

$$\int_{\tilde{\Omega}} \mu\,\nabla p \cdot \nabla\phi = \int_{\tilde{\Omega}} \mu\,\psi \cdot \nabla\phi\,. \tag{5.19}$$

(1) If the pair $(\mu,\tilde{\Omega})$ satisfies (5.16) and (5.17), and if $\psi \in [W^{1,2}(\tilde{\Omega})]^3$, then for $i = 0,\dots,m$, one has

$$p \in W^{2,2}(\tilde{\Omega}_i), \quad \|p\|_{W^{2,2}(\tilde{\Omega}_i)} \leq c\,\|\psi\|_{[W^{1,2}(\tilde{\Omega})]^3}\,.$$

(2) Suppose that $(\mu, \tilde{\Omega})$ satisfies (5.16) and (5.18). Then there exists some $q_1 > 3$ such that if $\psi \in [L^q(\tilde{\Omega})]^3$ for a $q \in]q_1', q_1[$, then

$$p \in W^{1,q}(\tilde{\Omega}), \quad \|p\|_{W^{1,q}(\tilde{\Omega})} \leq \bar{c}\,\|\psi\|_{[L^q(\tilde{\Omega})]^3} \,.$$

If $\partial\tilde{\Omega} \in \mathcal{C}^1$, then one may take $q_1 = +\infty$.

(3) If the number $1 - \mu_l/\mu_u$ is sufficiently small, then the same as in (2) is valid without further assumption on the function μ and on the domain $\tilde{\Omega}$.

Proof. The assertion (1) was proved in the paper Ladyzhenskaja and Solonnikov [1960], Lemma 1.

(2): Obviously, the functional

$$F(\zeta) := \int_{\tilde{\Omega}} \mu\, \psi \cdot \nabla \zeta \,,$$

is a well-defined element of $[W^{1,q'}(\tilde{\Omega})]^*$. Under our geometrical assumption on the domain $\tilde{\Omega}$, the remark 3.16 in Elschner et al. [2007] shows that the operator

$$\nabla \cdot (\mu\, \nabla) : W^{1,q}(\tilde{\Omega}) \longrightarrow [W^{1,q'}(\tilde{\Omega})]^* \,,$$

is a topological isomorphism. This proves the claim.

(3): In view of Lemma 5.3.1 (see the next section), there exists a constant C, depending on $\tilde{\Omega}$, such that if the smallness assumption $C\,(1 - \mu_l/\mu_u) < 1$ is satisfied, the assertion follows. $\square$

We need another auxiliary result concerning the possibility to find vector potentials in the space $V_{\mu,0}$.

Lemma 5.2.2. We consider a simply connected Lipschitz domain $\tilde{\Omega}$. Let $j \in L^2_{\text{div}}(\tilde{\Omega})$ be such that $\operatorname{div} j = 0$ in $\tilde{\Omega}$ in the generalized sense. Then we can find a vector potential $B \in V_{\mu,0}(\tilde{\Omega})$ such that

$$\operatorname{curl} B = j \,, \quad \|B\|_{V_\mu(\tilde{\Omega})} \leq c\,\|j\|_{[L^2(\tilde{\Omega})]^3} \,. \tag{5.20}$$

In addition, the following results are valid:

(1) If (5.16) and (5.17) are satisfied, then for $i = 0, \ldots, m$ we have

$$B \in [W^{1,2}(\tilde{\Omega}_i)]^3 \,, \quad \|B\|_{[W^{1,2}(\tilde{\Omega}_i)]^3} \leq c\,\|j\|_{[L^2(\tilde{\Omega})]^3} \,.$$

(2) If (5.16) and (5.18) are satisfied, or if the number $1 - \mu_l/\mu_u$ is sufficiently small, then there exists some $\tilde{\xi} > 3$ such that

$$B \in [L^{\tilde{\xi}}(\tilde{\Omega})]^3 \,, \quad \|B\|_{[L^{\tilde{\xi}}(\tilde{\Omega})]^3} \leq c\,\|j\|_{[L^2(\tilde{\Omega})]^3} \,.$$

If $\partial\tilde{\Omega} \in \mathcal{C}^1$, then we can choose $\tilde{\xi} = 6$.

Proof. By Lemma I.3.6 in Girault and Raviart [1986], we find a potential A in the space $L^2_{\mathrm{curl}}(\tilde{\Omega}) \cap L^2_{\mathrm{div}}(\tilde{\Omega})$ such that

$$\begin{aligned} &\operatorname{div} A = 0\,, \quad \operatorname{curl} A = j\,, \quad \text{in } \tilde{\Omega}\,, \\ &\gamma_n(A) = 0 \quad \text{on } \partial\tilde{\Omega}\,. \end{aligned} \tag{5.21}$$

In addition, there exists a positive constant C independent of j such that

$$\|A\|_{[L^2(\tilde{\Omega})]^3} \le C\, \|j\|_{[L^2(\tilde{\Omega})]^3}\,.$$

We consider the (up to constants) unique function $p \in W^{1,2}(\tilde{\Omega})$ that satisfies

$$\int_{\tilde{\Omega}} \mu\, \nabla p \cdot \nabla \phi = \int_{\tilde{\Omega}} \mu\, A \cdot \nabla \phi$$

for all $\phi \in W^{1,2}(\tilde{\Omega})$, and we set $B := A - \nabla p$. We verify easily that $B \in V_{\mu,0}(\tilde{\Omega})$, and that $\operatorname{curl} B = j$.

(1): If (5.16) and (5.17) are satisfied, then Theorem I.3.8 in Girault and Raviart [1986] even gives that the potential A in (5.21) belongs to $[W^{1,2}(\tilde{\Omega})]^3$, and that

$$\|A\|_{[W^{1,2}(\tilde{\Omega})]^3} \le \bar{c}\, \|j\|_{[L^2(\tilde{\Omega})]^3}\,.$$

Then, Lemma 5.2.1 implies that $p \in W^{2,2}(\tilde{\Omega}_i)$ for $i = 0, \ldots, m$, and that

$$\|p\|_{W^{2,2}(\tilde{\Omega}_i)} \le C\, \|A\|_{[W^{1,2}(\tilde{\Omega})]^3} \le \bar{C}\, \|j\|_{[L^2(\tilde{\Omega})]^3}\,.$$

Thus, B belongs to $[W^{1,2}(\tilde{\Omega}_i)]^3$ and satisfies the assertion.

(2): If $\partial\tilde{\Omega} \in \mathcal{C}^{0,1}$, we see that $A \in \mathcal{W}^{2,\infty}(\tilde{\Omega})$ (cp. (5.8)). Proposition 5.1.2 then implies the existence of a number $\xi > 3$ such that

$$\|A\|_{[L^\xi(\tilde{\Omega})]^3} \le \bar{c}\, \|j\|_{[L^2(\tilde{\Omega})]^3}\,.$$

If $\partial\tilde{\Omega} \in \mathcal{C}^1$, the choice $\xi = 6$ is possible. Under the hypothesis of the present lemma, it follows from Lemma 5.2.1 that there exists some $\tilde{\xi} \in]3, \xi]$ such that $p \in W^{1,\tilde{\xi}}(\tilde{\Omega})$, and that

$$\|p\|_{W^{1,\tilde{\xi}}(\tilde{\Omega})} \le C\, \|A\|_{[L^\xi(\tilde{\Omega})]^3} \le \hat{C}\, \|j\|_{[L^2(\tilde{\Omega})]^3}\,.$$

Therefore, $B = A + \nabla p$ belongs to $[L^{\tilde{\xi}}(\tilde{\Omega})]^3$ and satisfies the assertion. Again, if $\partial\tilde{\Omega} \in \mathcal{C}^1$, one can prove that the choice $\tilde{\xi} = 6$ is possible. □

Proposition 5.2.3. Let $\tilde{\Omega}$ be a simply connected Lipschitz domain .

(1) The validity of (5.16) and (5.17) for $(\mu, \tilde{\Omega})$, is sufficient for the topological identity

$$V_\mu(\tilde{\Omega}) = V_\mu(\tilde{\Omega}) \cap \bigcap_{i=0}^{m} [W^{1,2}(\tilde{\Omega}_i)]^3$$

(2) If the pair $(\mu, \tilde{\Omega})$ satisfies (5.16) and (5.18), then there exists a number $\tilde{\xi} > 3$ such that $V_\mu(\tilde{\Omega}) \hookrightarrow [L^{\tilde{\xi}}(\tilde{\Omega})]^3$ with continuous embedding. If $\partial\tilde{\Omega} \in \mathcal{C}^1$, then one can choose $\tilde{\xi} = 6$.

(3) If the number $1 - \mu_l/\mu_u$ is sufficiently small, then the same as in (2) is valid, without further assumption on the function μ and on the domain $\tilde{\Omega}$.

Proof. We consider an arbitrary $\psi \in V_\mu(\tilde{\Omega})$. Since *curl* ψ is divergence-free almost everywhere in $\tilde{\Omega}$, we find by Lemma 5.2.2 a $B \in V_{\mu,0}(\tilde{\Omega})$ such that

$$\operatorname{curl} B = \operatorname{curl} \psi \quad \text{ in } \tilde{\Omega}\,.$$

Since $\tilde{\Omega}$ is simply connected, we conclude from the fact that *curl* $(\psi - B) = 0$ that

$$\psi = B + \nabla p\,, \tag{5.22}$$

for some $p \in W^{1,2}(\tilde{\Omega})$. The function p is a weak solution to the transmission problem

$$\int_{\tilde{\Omega}} \mu \, \nabla p \cdot \nabla \phi = - \int_{\tilde{\Omega}} \operatorname{div}(\mu \, \psi) \, \phi + \sum_{i=0}^{m} \int_{\partial\tilde{\Omega}_i} \Big[\mu \, \psi \cdot \vec{n}\Big] \, \phi\,,$$

for all $\phi \in W^{1,2}(\tilde{\Omega})$. (1): Suppose that (5.16) and (5.17) are satisfied. Then it is shown in Ladyzhenskaja and Solonnikov [1960], Lemma 1, that $p \in W^{2,2}(\tilde{\Omega}_i)$ for $i = 0, \ldots, m$, and that

$$\|p\|_{W^{2,2}(\tilde{\Omega}_i)} \le c \, \| \operatorname{div}(\mu \, \psi) \|_{L^2(\tilde{\Omega})}\,.$$

Since by Lemma 5.2.2, we know that $B \in [W^{1,2}(\tilde{\Omega}_i)]^3$ for $i = 0, \ldots, m$, we obtain from (5.22) the norm estimate

$$\|\psi\|_{[W^{1,2}(\tilde{\Omega}_i)]^3} \le \|p\|_{W^{2,2}(\tilde{\Omega}_i)} + \|B\|_{[W^{1,2}(\tilde{\Omega}_i)]^3} \le c \, (\| \operatorname{div}(\mu \, \psi)\|_{L^2(\tilde{\Omega})} + \| \operatorname{curl} \psi\|_{[L^2(\tilde{\Omega})]^3})\,.$$

(2): If (5.16) and (5.18) are satisfied, resp. if the number $1 - \mu_l/\mu_u$ is sufficiently small, we can use the arguments of Elschner et al. [2007], resp. of the appendix, to obtain the existence of some $\tilde{\xi} > 3$ such that $p \in W^{1,\tilde{\xi}}(\tilde{\Omega})$. In addition, we find the norm estimate

$$\|p\|_{W^{1,\tilde{\xi}}(\tilde{\Omega})} \le c \, \| \operatorname{div}(\mu \, \psi) \|_{L^2(\tilde{\Omega})}\,.$$

Using Lemma 5.2.2 and (5.22), the claim follows. □

5.3 Relationship to the Helmholtz decomposition of L^q

In this appendix, we prove the auxiliary result needed for the proof of Proposition 5.2.3, (3).

Let $U \subset \mathbb{R}^3$ be a simply connected Lipschitz domain. Let $1 < q < \infty$ and $g \in [L^q(U)]^3$ be given. We assume that the function $\mu : U \to \mathbb{R}$ is measurable and satisfies (c. p. (3.44))

$$0 < \mu_l \leq \mu(x) \leq \mu_u < \infty \text{ for all } x \in U . \tag{5.23}$$

We consider the problem of finding $p \in W^{1,q}(U)$ that satisfies the integral relation

$$\int_U \mu \nabla p \cdot \nabla \phi = \int_U g \cdot \nabla \phi , \tag{5.24}$$

for all $\phi \in W^{1,q'}(U)$.

Lemma 5.3.1. Assume that the domain U is Lipschitzian, and let $1 < q < \infty$. Let μ be a measurable function that satisfies (5.23). Then, there exists some $q_1 > 3$ such that for all $q \in]q_1', q_1[$, we can find a positive constant $C = C(U, q)$, so that if the property

$$C \left(1 - \mu_l/\mu_u\right) < 1 \tag{5.25}$$

is satisfied, then the problem (5.24) possesses a (up to constants) unique solution $p \in W^{1,q}(U)$, and the estimate

$$\left(1 - C \left(1 - \mu_l/\mu_u\right)\right) \|\nabla p\|_{[L^q(U)]^3} \leq \|g\|_{[L^q(U)]^3}$$

is valid.

Proof. We use the arguments that we have learned from our coauthor in Druet et al. [2008]. In view of Proposition 5.3.5 below, there exists some $q_1 > 3$ such that for all $q \in]q_1', q_1[$ and $u \in [L^q(U)]^3$, there exists a (up to constants) unique $\zeta \in W^{1,q}(U)$ such that

$$\int_U \nabla \zeta \cdot \nabla \phi = \int_U u \cdot \nabla \phi ,$$

for all $\phi \in W^{1,q'}(U)$. In addition, the solution ζ satisfies a continuous estimate

$$\|\nabla \zeta\|_{[L^q(U)]^3} \leq C \|u\|_{[L^q(U)]^3} .$$

For $w \in W^{1,q}(U)$ arbitrary, we can thus find a unique $\zeta \in W^{1,q}(U)$ such that

$$\int_U \nabla \zeta \cdot \nabla \phi = \int_U \left(1 - \frac{\mu}{\mu_u}\right) \nabla w \cdot \nabla \phi + \int_U \frac{1}{\mu_u} g \cdot \nabla \phi , \tag{5.26}$$

for all $\phi \in W^{1,q'}(U)$. In addition, in view of Propostion 5.3.5, we find the norm estimate

$$\|\nabla \zeta\|_{W^{1,q}(U)} \leq C \left(1 - \frac{\mu_l}{\mu_u}\right) \|\nabla w\|_{[L^q(U)]^3} + \frac{1}{\mu_u} C \|g\|_{[L^q(U)]^3} . \tag{5.27}$$

We define the space $W^{1,q}_M(U)$ as the closed subspace of $W^{1,q}(U)$ that contains the functions with vanishing mean value over U. This space is a Banach space with respect to the norm $\|u\|_{W^{1,q}_M(U)} := \|\nabla u\|_{[L^q(U)]^3}$. We define a mapping $\mathcal{T} : W^{1,q}_M(U) \to W^{1,q}_M(U)$ by setting

$\mathcal{T}(w) := \zeta$, where ζ satisfies (5.26). From existence and unicity of ζ, we conclude that $\mathcal{T}$ is well defined. We prove easily that $\mathcal{T}$ is strictly contractive. As a matter of fact, we can write

$$\begin{aligned}\|\mathcal{T}(w_1) - \mathcal{T}(w_2)\|_{W^{1,q}_M(U)} &\le C \sup_{\|\nabla\phi\|_{[L^{q'}(U)]^3}\le 1} \left| \int_U \left(1 - \frac{\mu}{\mu_u}\right) \nabla(w_1 - w_2)\cdot\nabla\phi \right| \\ &\le C\left(1 - \frac{\mu_l}{\mu_u}\right) \|w_1 - w_2\|_{W^{1,q}_M(U)} \,. \qquad (5.28)\end{aligned}$$

Now, the Banach fixed point theorem proves the existence of a unique fixed point of $\mathcal{T}$. In view of (5.26), this fixed point is the unique solution of (5.24). □

In order to interpret the assumption (5.25), it would be interesting to know the exact dependence both on the domain U and on the exponent q of the constant C appearing in Lemma 5.3.1. Still having at this time to restrict ourselves to qualitative considerations, we can make the statement somewhat more precise. To this aim, we first recall some well-known notions. Let $U \subset \mathbb{R}^3$ be a bounded domain. We define

$$\mathcal{D}(U) := \left\{ \phi \in [C^\infty_c(U)]^3 \,\middle|\, \operatorname{div}\phi = 0 \text{ in } U \right\}.$$

For $1 < q < \infty$, we introduce closed subspaces $H_q(U)$, $G_q(U)$ of $[L^q(U)]^3$ defined by

$$H_q(U) := \text{ closure of } \mathcal{D}(U) \text{ in the norm } \|\cdot\|_{[L^q(U)]^3}\,,$$

$$G_q(U) := \left\{ \phi \in [L^q(U)]^3 \,\middle|\, \phi = \nabla\zeta,\, \zeta \in W^{1,q}(U) \right\}.$$

Definition 5.3.2. If the decomposition

$$[L^q(U)]^3 = H_q(U) \oplus G_q(U)\,, \qquad (5.29)$$

is valid, it is called *Helmholtz-Weyl decomposition* of the space $[L^q(U)]^3$.

Lemma 5.3.3. Let the assumptions of Lemma 5.3.1 be satisfied. Then, for all $q \in]q_1', q_1[$, the Helmholtz-Weyl decomposition of $[L^q(U)]^3$ is valid, and the smallest constant C for which (5.25) holds is given by $C = \|P_{Gq}\|_{\mathcal{L}([L^q(U)]^3,\,[L^q(U)]^3)}$, where P_{Gq} is the projection onto the space $G_q(U)$.

Proof. The validity of (5.29) in the range $]q_1', q_1[$ follows from the equivalent characterization of the Helmholtz-Weyl decomposition recalled in Lemma 5.3.4 below, and of Theorem 1.6 in Zanger [2000].

If the decomposition (5.29) is valid in $[L^q(U)]^3$, then one can show that for all $p \in W^{1,q}(U)$

$$\|\nabla p\|_{[L^q(U)]^3} \le \|P_{Gq}\|_{\mathcal{L}([L^q(U)]^3,\,[L^q(U)]^3)} \sup_{\|\nabla\phi\|_{[L^{q'}(U)]^3}\le 1} \left| \int_U \nabla p\cdot\nabla\phi \right|.$$

This was proved for example in Simader and Sohr [1992], Th. 1.3., Th. 6.1. Applying this result to estimate (5.28), the claim follows. □

The following equivalent characterization is well known.

Lemma 5.3.4. The following statements are equivalent.

(1) The Helmholtz-Weyl decomposition of the space $[L^q(U)]^3$ is valid.

(2) For $u \in [L^q(U)]^3$, there exists an (up to constants) unique $\zeta \in W^{1,q}(U)$ such that

$$\int_U \nabla\zeta \cdot \nabla\phi = \int_U u \cdot \nabla\phi \,,$$

for all $\phi \in W^{1,q'}(U)$.

Proof. See Galdi [1994], III. 1, Lemma 1.2, or Simader and Sohr [1992], Th. 6.1. □

Assume that $U \subset \mathbb{R}^3$ is a bounded Lipschitz domain. For numbers $1 < q < \infty$, we denote by $q' := q/(q-1)$ the conjugated exponent. For some linear functional $F \in [W^{1,q'}(U)]^*$ such that $F(1) = 0$, we consider the variational problem to find a function $w \in W^{1,q}(\Omega)$ such that

$$\int_U \nabla w \cdot \nabla\phi = F(\phi) \,, \tag{5.30}$$

for all $\phi \in W^{1,q'}(U)]^*$. Thanks to the results of the paper Zanger [2000], we can state a very general result on the solvability of (5.30).

Proposition 5.3.5. Assume that $U \subset \mathbb{R}^3$ is a bounded Lipschitz domain. Then, there exists some number $3 < q_1 < \infty$, such that for all $q \in]q_1', q_1[$, the problem (5.30) possesses an up to a constant unique weak solution $w \in W^{1,q}(U)$ whenever the right-hand side F belongs to $[W^{1,q'}(U)]^*$ and satisfies $F(1) = 0$. In addition, the estimate

$$\|w\|_{W^{1,q}(U)} \leq C\,\|F\|_{[W^{1,q'}(U)]^*} \,,$$

is valid.

Proof. We apply Theorem 1.6 in Zanger [2000] with $\alpha = 1$ therein. □

Chapter 6

The boundary value problem for the stationary system

The results of this section have been published in Druet [2008b].

In this chapter, we investigate the boundary value problem of paragraph 3.1.1. It is shown that the auxiliary results of chapter 4 and 5 are sufficient for a solution theory of (P_{st}).

6.1 Existence results

We recall that the domain $\tilde{\Omega}$ is assumed to be simply connected and Lipschitzian. Throughout this section, we will consider domains of the form $\overline{\Omega} = \bigcup_{i=0}^{m} \overline{\tilde{\Omega}_i}$ described in the paragraph 3.1. As we have shown in the section 5, additional assumptions on the pair of data μ and $\tilde{\Omega}$ may be necessary to obtain the regularity of the magnetic field required by the condition (3.65). We will in particular discuss the hypothesis (5.16) and (5.17), (5.18).

By the remarks of the paragraph (3.1.1), the buoyant forces $f = f(\theta)$ can be assumed to have the form (3.12). The term $\alpha\,(\theta - \theta_M)$ represents the magnitude of the density variations in the fluid. This quantity has to remain small compared to unity for the Boussinesq model to make sense. Throughout the present section, we replace the force term f by

$$f = -\rho_1\,\vec{g}\,\text{sign}(\theta - \theta_M)\,\min\{\alpha\,|\theta - \theta_M|,\, M_\theta\}\,, \tag{6.1}$$

with a positive number M_θ which can be interpreted as the maximal allowed density variations. In this section, we thus have have the global bound

$$\max_{\mathbb{R}} |f| \leq \rho_1\,|\vec{g}|\,M_\theta < \infty\,. \tag{6.2}$$

The nontruncated case (3.12) will be treated in the next section.

We introduce some notations. We denote by c_{Korn} the smallest positive constant such that for all $v \in D_0^{1,2}(\Omega_1)$,

$$\int_{\Omega_1} |\nabla v|^2 \leq c_{\text{Korn}} \int_{\Omega_1} D(v\,,\,v)\,.$$

For Lipschitz domains, it is obvious that $c_{\text{Korn}} = 2$. We denote by $c_{\mathcal{H}_\mu} > 0$ a constant such that

$$\|\psi\|^2_{\mathcal{H}_\mu(\tilde{\Omega})} \leq c_{\mathcal{H}_\mu} \int_{\tilde{\Omega}} |\operatorname{curl}\psi|^2 \,,$$

whose existence is granted in view of Lemma A.4.1. The constant $c_{\mathcal{H}}$ can be estimated by $c_{\mathcal{H}_\mu} \leq c_{\mathcal{H}}\, \mu_u/\mu_l$ (see Lemma A.4.1, (1)), where $c_{\mathcal{H}}$ is the corresponding constant for $\mu \equiv 1$. In our estimates, we will use the abbreviations

$$\mathbf{v_g} \,:=\, \max_{\partial\Omega_1} |v_g| \,, \quad L \,:=\, \operatorname{diam}(\Omega_1)\,.$$

Through this section, we also suppose that v_g is given by (3.39) and satisfies the smallness assumption

$$\mathbf{v_g} < c \, \min\left\{ \frac{\eta_l}{\rho_1\, L} \,,\, \frac{r_l\,\sqrt{\mu_l}}{\mu_u^{3/2}} \right\} \,, \tag{6.3}$$

with $c := \min\{c_{\text{Korn}}^{-1}\,,\, c_{\mathcal{H}}^{-1}\}$. If (6.3) is valid, we can introduce the positive number

$$\gamma_0 := \min\{\eta_l \,-\, c_{\text{Korn}}\,\rho_1\,\mathbf{v_g}\,L\,,\, r_l \,-\, 2\,c_{\mathcal{H}}\,\mu_u \mathbf{v_g}\}\,. \tag{6.4}$$

Our main result in this section is the following theorem.

Theorem 6.1.1. Assume that $\tilde{\Omega}$ is a simply connected Lipschitz domain that has the structure described in the paragraph 3.1. Let the assumptions of section 3.1.3 for the data be valid, with either (3.50) or (3.51). Assume in addition that the surface Σ belongs to $\mathcal{C}^{1,\alpha}$ for some $\alpha > 0$.

Assume finally for the boundary data that $v_g \in D^{1,2}(\Omega_1) \cap L^\infty(\Omega_1)$ satisfies the smallness assumption (6.3), and that the imposed temperature θ_g belongs to $W^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Then, if the force f has the form (6.1), there exists at least one weak solution of (P_{st}) in the sense of Definition 3.2.1.

The rest of the section is devoted to the proof of Theorem 6.1.1. We at first need to introduce some additional notations. For vector fields $v \in D^{1,2}_0(\Omega_1)$, we use the notation

$$\hat{v} \,:=\, v + v_g\,. \tag{6.5}$$

Thanks to the assumption (3.49), we can fix a $\phi_0 \in C^\infty(\overline{\Omega})$ such that $\phi_0 = 1$ on Γ and $\phi_0 = 0$ on Σ. For $\theta \in V^{2,5}_\Gamma(\Omega)$, we introduce the notation

$$\hat{\theta} \,:=\, \theta + \theta_g\,\phi_0\,. \tag{6.6}$$

In this way, we homogenize the problem for the temperature without perturbing the nonlocal terms on Σ. Given a current density j_0 with (3.42), we can find by Lemma 5.2.2 some $H_0 \in \mathcal{H}_\mu(\tilde{\Omega})$ such that

$$\operatorname{curl} H_0 = j_0 \;\text{ in } \tilde{\Omega}\,. \tag{6.7}$$

For vector fields $H \in \mathcal{H}^0_\mu(\tilde{\Omega})$, we then define a reaction field

$$\hat{H} := H + H_0\,. \tag{6.8}$$

For a function $g : \tilde{\Omega} \longrightarrow \mathbb{R}$ and $\delta \in \mathbb{R}^+$, we introduce the cutoff

$$[g]_{(\delta)} := \frac{g}{1 + \delta\,|g|}\,. \tag{6.9}$$

In the next propostion, we construct approximate solutions by penalizing the heat sources on the right-hand side of the energy equation.

Proposition 6.1.2. Let $\delta > 0$ be an arbitrary positive number. If the assumptions of Theorem 6.1.1 are satisfied, there exists a triple

$$\{v,\, H,\, \theta\} \in D^{1,2}(\Omega_1) \times \mathcal{H}_\mu(\tilde{\Omega}) \times V^{2,5}(\Omega)\,,$$

such that $v = v_g$ on $\partial\Omega_1$, $\theta = \theta_g$ on Γ, $\operatorname{curl} H = j_0$ in $\tilde{\Omega}_{c_0}$ and such that the relations

$$\int_{\Omega_1} \rho_1\,(v\cdot\nabla)v\cdot\phi + \int_{\Omega_1} \eta(\theta)\,D(v\,,\,\phi) = \int_{\Omega_1} \big(\operatorname{curl} H \times \mu\, H\big)\cdot\phi + \int_{\Omega_1} f(\theta)\cdot\phi\,, \tag{6.10}$$

$$\int_{\tilde{\Omega}} r(\theta)\ \operatorname{curl} H \cdot \operatorname{curl}\psi = \int_{\Omega_1} \big(v \times \mu\, H\big)\cdot\ \operatorname{curl}\psi, \tag{6.11}$$

$$\begin{aligned} &\int_{\Omega_1} \rho_1\, c_V\, v\cdot\nabla\theta\,\xi + \int_{\Omega} \kappa(\theta)\,\nabla\theta\cdot\,\nabla\xi + \int_{\Sigma} G(\sigma\,\theta^4)\,\xi \\ &\qquad = \int_{\Omega} \Big[r(\theta)\,|\operatorname{curl} H|^2 + \chi_{\Omega_1}\,\eta(\theta)\,D(v\,,\,v)\Big]_{(\delta)}\,\xi\,, \end{aligned} \tag{6.12}$$

are satisfied for all $\{\phi,\,\psi,\,\xi\} \in D^{1,2}_0(\Omega_1) \times \mathcal{H}^0_\mu(\tilde{\Omega}) \times V^{2,5}_\Gamma(\Omega)$. In addition,

$$\theta \geq \operatorname*{ess\,inf}_{\Gamma} \theta_g \qquad \text{almost everywhere in } \Omega\,. \tag{6.13}$$

Proof. Note that the right-hand side of the heat equation is not a compact perturbation of a monotone operator. Proofs of similar results have though become fairly standard (cp. Naumann [2005]) thanks to the theory of pseudomonotone operators. The main interest in the present case consists in the handling of the Joule heating term and the nonlocal radiation terms. Define

$$V := D^{1,2}_0(\Omega_1) \times \mathcal{H}^0_\mu(\tilde{\Omega}) \times V^{2,5}_\Gamma(\Omega)\,,$$

which is separable and reflexive. Then, the isomorphy

$$V^* \cong [D^{1,2}_0(\Omega_1)]^* \times [\mathcal{H}^0_\mu(\tilde{\Omega})]^* \times [V^{2,5}_\Gamma(\Omega)]^*$$

is valid. Throughout this proof, we denote by $\langle\cdot,\cdot\rangle$ the duality pairing between V and V^*.

Recalling the notations (6.5), (6.6), (6.8) and (6.9), we define an operator $A : V \to V^*$ by

$$\begin{aligned}
&\big\langle A(\{v,H,\theta\})\,,\,\{\phi,\psi,\xi\}\big\rangle \\
&:= \int_{\Omega_1} \rho_1\,(\hat v\cdot\nabla)\hat v\cdot\phi + \int_{\Omega_1}\eta(\hat\theta)\,D(\hat v\,,\,\phi) - \int_{\Omega_1}\big(\operatorname{curl}\hat H\times\mu\,\hat H\big)\cdot\phi \\
&\quad - \int_{\Omega_1} f(\hat\theta)\cdot\phi + \int_{\tilde\Omega} r(\hat\theta)\,\operatorname{curl}\hat H\cdot\operatorname{curl}\psi - \int_{\Omega_1}\big(\hat v\times\mu\,\hat H\big)\cdot\operatorname{curl}\psi + \int_{\Omega_1}\rho_1\,c_V\,\hat v\cdot\nabla\hat\theta\,\xi \\
&\quad + \int_\Omega \kappa(\hat\theta)\,\nabla\hat\theta\cdot\nabla\xi + \int_\Sigma G(\sigma\,|\hat\theta|^3\,\hat\theta)\,\xi - \int_\Omega\Big[r(\hat\theta)\,|\operatorname{curl}\hat H|^2 + \chi_{\Omega_1}\,\eta(\hat\theta)\,D(\hat v\,,\,\hat v)\Big]_{(\delta)}\xi\,.
\end{aligned}$$

Note that using the results of Proposition 5.2.3 we have under the assumption (3.50) or (3.51) for $i = 0,\ldots,m$ the continuous embedding

$$\mathcal{H}_\mu(\tilde\Omega) \hookrightarrow [L^q(\tilde\Omega)]^3\,, \tag{6.14}$$

with $q > 3$. We can therefore prove under the assumptions of Theorem 6.1.1 that A is well defined.

We now want to prove the result using the well-known fact that the coercivity and the pseudomonotonicity of the operator A are sufficient for its surjectivity. We first discuss the coercivity. Observe that

$$\int_{\Omega_1}\rho_1\,(\hat v\cdot\nabla)v\cdot v = \int_{\Omega_1}\rho_1\,\hat v_j\,\frac{1}{2}\,\frac{\partial}{\partial x_j}v_i^2 = 0\,,$$

$$\int_{\Omega_1}\rho_1\,c_V\,\hat v\cdot\nabla\theta\,\theta = \int_{\Omega_1}\rho_1\,c_V\,\hat v_j\,\frac{1}{2}\,\frac{\partial}{\partial x_j}\theta^2 = 0\,,$$

since $v\in D_0^{1,2}(\Omega_1)$, and since v_g is divergence free in Ω_1 and tangential on $\partial\Omega_1$. It follows that

$$\begin{aligned}
\int_{\Omega_1}\rho_1\,(\hat v\cdot\nabla)\hat v\cdot v &= \int_{\Omega_1}\rho_1\,\big((v\cdot\nabla)v_g + (v_g\cdot\nabla)v_g\big)\cdot v \\
&= -\int_{\Omega_1}\rho_1\,(v_j\,v_{g,i} + v_{g,i}\,v_{g,j})\,\frac{\partial v_i}{\partial x_j}\,.
\end{aligned} \tag{6.15}$$

Thus, by Poincaré's and Young's inequality, we find the estimate

$$\begin{aligned}
\left|\int_{\Omega_1}\rho_1\,(\hat v\cdot\nabla)\hat v\cdot v\right| &\le \rho_1\,\Big(\mathbf{v_g}\,L\,\|\nabla v\|_{[L^2(\Omega_1)]^9} + \mathbf{v_g}^2\,\operatorname{meas}(\Omega_1)^{1/2}\Big)\,\|\nabla v\|_{[L^2(\Omega_1)]^9} \\
&\le (\rho_1\,\mathbf{v_g}\,L + \gamma)\,\|\nabla v\|^2_{[L^2(\Omega_1)]^9} + \frac{\rho_1^2\,\mathbf{v_g}^4\,\operatorname{meas}(\Omega_1)}{4\,\gamma}\,,
\end{aligned}$$

where γ is an arbitrary small, positive number. We consider also the estimate

$$\begin{aligned}
\left|\int_{\Omega_1}\big(v_g\times\mu\,\hat H\big)\cdot\operatorname{curl}H\right| &\le 2\,\mathbf{v_g}\,\mu_u\,\|H+H_0\|_{[L^2(\Omega_1)]^3}\,\|\operatorname{curl}H\|_{[L^2(\Omega_1)]^3} \\
&\le (2\,\mathbf{v_g}\,\mu_u\,c_{\mathcal{H}} + \gamma)\,\|\operatorname{curl}H\|^2_{[L^2(\Omega_1)]^3} + \frac{\mathbf{v_g}^2\,\mu_u^2}{\gamma}\,\|H_0\|^2_{[L^2(\Omega_1)]^3}\,.
\end{aligned}$$

Therefore, we can at first write

$$
\begin{aligned}
&\left\langle A\big(\{v,H,\theta\}\big)\,,\,\{v,H,\theta\}\right\rangle \\
&\quad\geq \int_{\Omega_1}\eta(\hat\theta)\,D(\hat v\,,\,v)-\int_{\Omega_1}\big(\operatorname{curl}\hat H\times\mu\,\hat H\big)\cdot v-\int_{\Omega_1}f(\hat\theta)\cdot v+\int_{\tilde\Omega}r(\hat\theta)\,\operatorname{curl}\hat H\cdot\operatorname{curl}H \\
&\quad-\int_{\tilde\Omega}\big(v\times\mu\,\hat H\big)\cdot\,\operatorname{curl}H+\int_{\Omega}\kappa(\hat\theta)\nabla\hat\theta\cdot\nabla\theta+\int_{\Sigma}G(\sigma\,|\hat\theta|^3\,\hat\theta)\,\theta \\
&\quad-\int_{\Omega}\Big[r(\hat\theta)\,|\operatorname{curl}\hat H|^2+\chi_{\Omega_1}\,\eta(\hat\theta)\,D(\hat v\,,\,\hat v)\Big]_{(\delta)}\theta \\
&\quad-(\rho_1\,\mathbf{v_g}\,L+\gamma)\,\|\nabla v\|^2_{[L^2(\Omega_1)]^9}-(2\,\mathbf{v_g}\,\mu_u\,c_{\mathcal H}+\gamma\,)\,\|\operatorname{curl}H\|^2_{[L^2(\Omega_1)]^3}-C_\gamma\,.
\end{aligned}\tag{6.16}
$$

where the precise value of the constant C_γ is not anymore needed. Further we observe that

$$
\int_{\Omega_1}\big(v\times\mu\,\hat H\big)\cdot\operatorname{curl}H=-\int_{\Omega_1}\big(\operatorname{curl}H\times\mu\,\hat H\big)\cdot v=-\int_{\Omega_1}\big(\operatorname{curl}\hat H\times\mu\,\hat H\big)\cdot v\,,\tag{6.17}
$$

since $curl\,H_0=j_0=0$ in Ω_1.

By the homogenization (6.6) and the coercivity result of Lemma 4.3.2, we have on the other hand that

$$
\begin{aligned}
&\int_{\Omega}\kappa(\hat\theta)\,\nabla\hat\theta\cdot\nabla\theta+\int_{\Sigma}G(\sigma\,|\hat\theta|^3\,\hat\theta)\,\theta \\
&\qquad=\int_{\Omega}\kappa(\hat\theta)\,|\nabla\theta|^2+\int_{\Sigma}G(\sigma\,|\theta|^3\,\theta)\,\theta-\int_{\Omega}\kappa(\hat\theta)\,\nabla(\theta_0\,\phi_0)\cdot\nabla\theta \\
&\qquad\geq c\,\min\Big\{\|\theta\|^2_{V^{2,5}_\Gamma(\Omega)}\,,\,\|\theta\|^5_{V^{2,5}_\Gamma(\Omega)}\Big\}-\int_{\Omega}\kappa(\hat\theta)\,\nabla(\theta_g\,\phi_0)\cdot\nabla\theta\,.
\end{aligned}
$$

By Young's inequality, this implies that

$$
\begin{aligned}
&\int_{\Omega}\kappa(\hat\theta)\,\nabla\hat\theta\cdot\nabla\theta+\int_{\Sigma}G(\sigma\,|\hat\theta|^3\,\hat\theta)\,\theta \\
&\qquad\geq c\,\min\Big\{\|\theta\|^2_{V^{2,5}_\Gamma(\Omega)}\,,\,\|\theta\|^5_{V^{2,5}_\Gamma(\Omega)}\Big\}-\gamma\,\|\nabla\theta\|^2_{L^2(\Omega)}\,-c_\gamma\,\|\nabla\theta_g\|^2_{L^2(\Omega)} \\
&\qquad\geq\bar c\,\|\theta\|^2_{V^{2,5}_\Gamma(\Omega)}-C\,.
\end{aligned}
$$

If we additionally take into account the facts

$$
\begin{aligned}
\left|\int_{\Omega_1}f(\hat\theta)\cdot v\right|&\leq\rho_1\,|\vec g|\,M_\theta\,\|v\|_{[L^1(\Omega_1)]^3}\,,\\
\left|\int_{\Omega}\Big[r(\hat\theta)\,|\operatorname{curl}\hat H|^2+\chi_{\Omega_1}\,\eta(\hat\theta)\,D(\hat v\,,\,\hat v)\Big]_{(\delta)}\theta\right|&\leq\frac{1}{\delta}\,\|\theta\|_{L^1(\Omega_1)}\,,
\end{aligned}
$$

we find by (6.16) and Young's inequality that

$$\left\langle A\big(\{v,H,\theta\}\big)\,,\,\{v,H,\theta\}\right\rangle \;\geq \frac{\gamma_0}{2}\,\|\{v,H,\theta\}\|_V^2 - C\,,$$

with the number γ_0 given by (6.4). This proves the coercivity.

In order to prove that A is pseudomonotone, we consider an arbitrary sequence such that

$$\begin{aligned}&\{v_k,H_k,\theta_k\}\subset V\\ &v_k \rightharpoonup v \text{ in } D_0^{1,2}(\Omega_1)\,,\qquad H_k \rightharpoonup H \text{ in } \mathcal{H}^0_\mu(\tilde\Omega)\,,\qquad \theta_k \rightharpoonup \theta \text{ in } V_\Gamma^{2,5}(\Omega)\,,\end{aligned}\tag{6.18}$$

and we assume that

$$\limsup_{k\to\infty}\left\langle A\big(\{v_k,H_k,\theta_k\}\big)\,,\,\{v_k,H_k,\theta_k\}-\{v,H,\theta\}\right\rangle \leq 0\,.\tag{6.19}$$

By well-known compactness properties and Lemma A.4.1, , we find a subsequence, that we not relabel, such that

$$\begin{aligned}&v_k \longrightarrow v \text{ in } L^4(\Omega_1)\,,\qquad H_k \longrightarrow H \text{ in } L^2(\tilde\Omega)\,,\\ &\theta_k \longrightarrow \theta \text{ in } L^2(\Omega)\,,\qquad \theta_k \longrightarrow \theta \text{ in } L^2(\Sigma)\,,\qquad \theta_k \longrightarrow \theta \text{ almost everywhere in } \Omega\,.\end{aligned}\tag{6.20}$$

Using straightforward rearrangements of terms we can write

$$\begin{aligned}&\int_{\Omega_1}\eta(\hat\theta_k)\,D(v_k-v\,,\,v_k-v)+\int_{\tilde\Omega}r(\hat\theta_k)\,|\,\mathrm{curl}(H_k-H)|^2+\int_\Omega\kappa(\hat\theta_k)\,|\nabla(\theta_k-\theta)|^2\\ &=\left\langle A\big(\{v_k,H_k,\theta_k\}\big)\,,\,\{v_k,H_k,\theta_k\}-\{v,H,\theta\}\right\rangle-\int_{\Omega_1}\eta(\hat\theta_k)\,D(\hat v\,,\,v_k-v)\\ &-\int_{\tilde\Omega}r(\hat\theta_k)\,\mathrm{curl}\hat H\cdot\mathrm{curl}(H_k-H)-\int_\Omega\kappa(\hat\theta_k)\,\nabla\hat\theta\cdot\nabla(\theta_k-\theta)\\ &\quad-\int_{\Omega_1}\rho_1\,(\hat v_k\cdot\nabla)\hat v_k\cdot(v_k-v)+\int_{\Omega_1}\big(\mathrm{curl}\,\hat H_k\times\mu\,\hat H_k\big)\cdot(v_k-v)\\ &+\int_{\Omega_1}f(\hat\theta_k)\cdot(v_k-v)+\int_{\Omega_1}\big(\hat v_k\times\mu\,\hat H_k\big)\cdot\mathrm{curl}(H_k-H)\\ &-\int_\Omega\rho\,c_V\,\hat v_k\cdot\nabla\hat\theta_k\,(\theta_k-\theta)-\int_\Sigma G(\sigma\,|\hat\theta_k|^3\,\hat\theta_k)\,(\theta_k-\theta)\\ &\quad+\int_\Omega\Big[r(\hat\theta_k)\,|\,\mathrm{curl}\,\hat H_k|^2+\chi_{\Omega_1}\,\eta(\hat\theta_k)\,D(v_k\,,\,v_k)\Big]_{(\delta)}(\theta_k-\theta)\,.\end{aligned}$$

Observe that

$$\begin{aligned}\int_\Sigma G(\sigma\,|\hat\theta_k|^3\,\hat\theta_k)\,(\theta_k-\theta)&=\int_\Sigma G(\sigma\,|\theta_k|^3\,\theta_k)\,(\theta_k-\theta)=\int_\Sigma\sigma\,|\theta_k|^3\,\theta_k\,G(\theta_k-\theta)\\ &=\int_\Sigma\epsilon\,\sigma\,|\theta_k|^3\,\theta_k\,(\theta_k-\theta)-\int_\Sigma\epsilon\,\sigma\,|\theta_k|^3\,\theta_k\,\tilde{\mathbf{H}}(\theta_k-\theta)\,,\end{aligned}$$

where the operator $\tilde{\mathbf{H}}$ is compact from $L^{5/4}(\Sigma)$ into itself according to Lemma 4.1.14, (1).Thus, passing to subsequences if necessary, we find that

$$\liminf_{k\to\infty}\int_\Sigma G(\sigma\,|\hat\theta_k|^3\,\hat\theta_k)\,(\theta_k-\theta)=\liminf_{k\to\infty}\int_\Sigma \epsilon\,\sigma\,|\theta_k|^3\,\theta_k\,(\theta_k-\theta)\geq 0\,. \tag{6.21}$$

By (6.19) and (6.20) and (6.21), we see immediately that

$$\limsup_{k\to\infty}\left(\int_{\Omega_1} D(v_k-v\,,\,v_k-v)+\int_{\tilde\Omega}|\operatorname{curl}(H_k-H)|^2+\int_\Omega|\nabla(\theta_k-\theta)|^2\right)\leq 0\,.$$

We thus find (not relabelled) subsequences with the properties

$$v_k\longrightarrow v \text{ in } D_0^{1,2}(\Omega_1)\,,\qquad H_k\longrightarrow H \text{ in } \mathcal{H}^0_\mu(\tilde\Omega)\,. \tag{6.22}$$

By the dominated convergence theorem this implies for a subsequence that

$$\Big[r(\hat\theta_k)\,|\operatorname{curl}\hat H_k|^2+\chi_{\Omega_1}\,\eta(\hat\theta_k)\,D(v_k\,,\,v_k)\Big]_{(\delta)}\longrightarrow\Big[r(\hat\theta)\,|\operatorname{curl}\hat H|^2+\chi_{\Omega_1}\,\eta(\hat\theta)\,D(v\,,\,v)\Big]_{(\delta)}\,,$$

strongly in $L^q(\Omega)$ for all $1\leq q<\infty$. We observe that by the compactness of the non local operator $\tilde{\mathbf{H}}$ and (6.18), we have generally

$$\liminf_{k\to\infty}\int_\Sigma G(\sigma\,|\theta_k|^3\,\theta_k)\,(\theta_k-\xi)\geq\int_\Sigma G(\sigma\,|\theta|^3\,\theta)\,(\theta-\xi)\,,$$

for all $\xi\in V^{2,5}_\Gamma(\Omega)$. By this property and (6.22), we easily can show that

$$\begin{aligned}&\liminf_{k\to\infty}\big\langle\, A\big(\{v_k,H_k,\theta_k\}\big)\,,\,\{v_k,H_k,\theta_k\}-\{\phi,\psi,\xi\}\,\big\rangle\\ &\qquad\geq\big\langle\, A\big(\{v,H,\theta\}\big)\,,\,\{v,H,\theta\}-\{\phi,\psi,\xi\}\,\big\rangle\,,\end{aligned}$$

for all $\{\phi,\psi,\xi\}\in V$, proving the preudomonotonicity of A. By the results of Lions [1969], Ch. 2, Th. 2.7., the equation $A\big(\{v,H,\theta\}\big)=0$ has at least one solution in V.

We now prove that (6.13) is valid. By the previous considerations, we have obtained in particular the relation

$$\begin{aligned}\int_{\Omega_1}\rho\,c_V\,\hat v\cdot\nabla\hat\theta\,\xi+\int_\Omega\kappa(\hat\theta)\,\nabla\hat\theta\cdot\nabla\xi+\int_\Sigma G(\sigma\,|\hat\theta|^3\,\hat\theta)\,\xi\qquad&\\ =\int_\Omega\Big[r(\hat\theta)\,|\operatorname{curl}\hat H|^2+\chi_{\Omega_1}\,\eta(\hat\theta)\,D(\hat v\,,\,\hat v)\Big]_{(\delta)}\,\xi\,,&\end{aligned} \tag{6.23}$$

for all $\xi\in V^{2,5}_\Gamma(\Omega)$. We define

$$k_0\;:=\;\operatorname*{ess\,inf}_\Gamma\theta_g\,, \tag{6.24}$$

and test with the function $\xi=(\hat\theta-k_0)^-$ in the relation (6.23). We observe that

$$\int_{\Omega_1}\rho\,c_V\,\hat v\cdot\nabla\hat\theta\,(\hat\theta-k_0)^-=\int_\Omega\rho\,c_V\,\hat v\cdot\frac12\nabla(\hat\theta-k_0)^{-^2}=0$$

$$\int_\Sigma G(\sigma\,|\hat\theta|^3\,\hat\theta)\,(\hat\theta-k_0)^-=\int_\Sigma G(\sigma\,|\hat\theta|^3\,\hat\theta)\,[(\hat\theta-k_0)^-+k_0]\geq 0\,.$$

Here, we used the fact that $G(1) = 0$ and the elementary properties of the operator G in enclosures.In order to obtain the inequality, we applied Lemma 4.3.6. We get $\int_\Omega \kappa(\hat\theta)\,\left|\nabla(\hat\theta - k_0)^-\right|^2 \le 0$, and since $\hat\theta \ge k_0$ on Γ, it follows that $\hat\theta \ge k_0$ almost everywhere in Ω. We can replace the term $|\hat\theta|^3\,\hat\theta$ by $\hat\theta^4$ in (6.23). We obtain (6.12). Writing from now on $\{v,\, H,\, \theta\}$ instead of $\{\hat v,\, \hat H,\, \hat\theta\}$, this finishes the proof of the proposition. □

For a sequence of approximate solutions according to Proposition 6.1.2, we introduce the notation

$$M_\delta := \frac{1}{\operatorname{meas}(\Sigma)} \int_\Sigma \theta_\delta^4\,. \tag{6.25}$$

Proposition 6.1.3. Any sequence of approximations $\{v_\delta,\, H_\delta,\, \theta_\delta\}$ according to Proposition 6.1.2 satisfies the following uniform estimates

(1) We can estimate the MHD energy by

$$\|\, v_\delta \,\|_{D^{1,2}(\Omega_1)} + \|\, H_\delta \,\|_{\mathcal{H}_\mu(\tilde\Omega)} \le c\,(\|f(\theta_\delta)\|_{[L^2(\Omega_1)]^3} + \|\, j_0 \,\|_{[L^2(\tilde\Omega_{c_0})]^3} + \|v_g\|_{D^{1,2}(\Omega_1)})$$

(2) The temperature is uniformly bounded by

$$\begin{aligned}&\|\, \theta_\delta \,\|_{W^{1,r}_\Gamma(\Omega)} + \|\, \theta_\delta^4 - M_\delta \,\|_{L^1(\Sigma)} \\ &\qquad \le P_r(\|f(\theta_\delta)\|_{[L^2(\Omega_1)]^3},\, \|\, j_0 \,\|_{[L^2(\tilde\Omega_{c_0})]^3},\, \|v_g\|_{D^{1,2}(\Omega_1)},\, \|\nabla\theta_g\|_{[L^2(\Omega)]^3},\, \|\theta_g\|_{L^\infty(\Omega)})\,,\end{aligned}$$

with a continuous function P_r for all $1 \le r < \frac{3}{2}$.

Proof. For the sake of notational simplicity, we write throughout this proof v instead of v_δ etc.

(1): We test in (6.10) with the field $v - v_g$, and in (6.11) with $H - H_0$. Recalling (6.15) and (6.17), we obtain, after adding both relations, that

$$\begin{aligned}\int_{\Omega_1} \eta(\theta)\, D(v\,,\, v) + \int_{\tilde\Omega} r(\theta)\,|\operatorname{curl} H|^2 &= \int_{\Omega_1} \rho_1\, v_j\, v_{g,i}\, \frac{\partial v_i}{\partial x_j} + \int_{\Omega_1} \eta(\theta)\, D(v\,,\, v_g) \\ &+ \int_{\Omega_1} f(\theta)\cdot(v - v_g) + \int_{\Omega_1} \big(v_g \times \mu\, H\big)\cdot \operatorname{curl} H + \int_{\tilde\Omega} r(\theta)\, j_0\cdot\operatorname{curl} H\,.\end{aligned} \tag{6.26}$$

By standard inequalities, we find for the absolute value of the right-hand side of (6.26) the upper bound

$$\begin{aligned}\int_{\Omega_1} \eta(\theta)\, D(v\,,\, v) + \int_{\tilde\Omega} r(\theta)\,|\operatorname{curl} H|^2 &\le \rho_1\, \mathbf{v_g}\, L\, \|\nabla v\|^2_{[L^2(\Omega_1)]^9} + \gamma \int_{\Omega_1} \eta(\theta)\, D(v\,,\, v) \\ &+ \gamma_2\, \|\nabla v\|^2_{[L^2(\Omega_1)]^9} + \frac{L^2\, \|f(\theta)\|^2_{[L^2(\Omega_1)]^3}}{4\,\gamma_2} + \frac{1}{4\gamma}\int_{\Omega_1} \eta(\theta)\, D(v_g\,,\, v_g) \\ &+ \|f(\theta)\cdot v_g\|_{[L^1(\Omega_1)]^3} + 2\,\mathbf{v_g}\,\mu_u\, c_{\mathcal{H}}\, \|\operatorname{curl} H\|^2_{[L^2(\Omega_1)]^3} + \gamma \int_{\tilde\Omega} r(\theta)\,|\operatorname{curl} H|^2 \\ &+ \frac{1}{4\gamma}\int_{\tilde\Omega} r(\theta)\,|j_0|^2\,,\end{aligned}$$

where γ, γ_2 are arbitrary small positive numbers. We obtain that

$$[(1-\gamma)\,\eta_l\, c_{\text{Korn}}^{-1} - \rho_1\,\mathbf{v_g}\,L - \gamma_2] \int_{\Omega_1} |\nabla v|^2 + [(1-\gamma)\,r_l - c_{\mathcal{H}}\,\mathbf{v_g}\,\mu_u] \int_{\tilde{\Omega}} |\operatorname{curl} H|^2$$

$$\leq \frac{L^2}{4\gamma_2}\,\|f(\theta)\|^2_{[L^2(\Omega_1)]^3} + \|f(\theta)\cdot v_g\|_{[L^1(\Omega_1)]^3} + \frac{1}{4\gamma}\int_{\tilde{\Omega}} r(\theta)\,|j_0|^2\,.$$

where we can choose γ, γ_2 arbitrary small, and the estimate (1) follows from the assumption (6.4).

(2): For a parameter $\gamma > 0$ to be fixed later, we introduce the continuous function

$$g_\gamma(t) := \operatorname{sign}(t)\,\left(1 - \frac{1}{(1+|t|)^\gamma}\right) \qquad \text{for } t \in \mathbb{R}\,.$$

In (6.12) we use the test function

$$\xi = \xi_\gamma :=\; g_\gamma(\theta - \tilde{\theta}_0) = \operatorname{sign}(\theta - \tilde{\theta}_0)\,\left(1 - \frac{1}{(1+|\theta-\tilde{\theta}_0|)^\gamma}\right)\,.$$

Here, we have set $\tilde{\theta}_0 := \theta_g\,\phi_0$, with a smooth function ϕ_0 such that $\phi_0 = 0$ on Σ and $\phi_0 = 1$ on Γ. Note that ξ vanishes on the boundary Γ, that $0 \leq \xi \leq 1$ in Ω, and that

$$\nabla\xi \;=\; \gamma\,\frac{\nabla(\theta-\tilde{\theta}_0)}{(1+|\theta-\tilde{\theta}_0|)^{1+\gamma}}\,,$$

so that we are allowed to test the relation (6.12) with this function.

Denoting by Ψ the primitive function of g_γ that vanishes at zero, we observe that

$$\int_{\Omega_1} \rho_1\,c_V\,v\cdot\nabla(\theta-\tilde{\theta}_0)\,\xi = \int_{\Omega} \rho_1\,c_V\,v\cdot\nabla\Psi(\theta-\tilde{\theta}_0) = 0\,.$$

By Lemma 4.3.6 and the fact that $\tilde{\theta}_0$ vanishes on Σ, we obtain on the other hand that

$$\int_\Sigma G(\sigma\,\theta^4)\,\xi = \int_\Sigma G(\sigma\,\theta^4)\,\left(1 - \frac{1}{(1+\theta)^\gamma}\right) \geq 0\,.$$

Thus, the inequality

$$\gamma\int_\Omega \frac{\kappa(\theta)\,|\nabla(\theta-\tilde{\theta}_0)|^2}{(1+|\theta-\tilde{\theta}_0|)^{1+\gamma}} \leq \gamma\int_\Omega \kappa(\theta)\frac{|\nabla\theta_0|\,|\nabla(\theta-\tilde{\theta}_0)|}{(1+|\theta-\tilde{\theta}_0|)^{1+\gamma}} + \int_{\Omega_1} \rho_1\,c_V\,|v\cdot\nabla\tilde{\theta}_0|$$

$$+ \int_\Omega \Big[r(\theta)\,|\operatorname{curl} H|^2 + \eta(\theta)\,D(v\,,\,v)\Big]_{(\delta)}\,,$$

is readily verified. By Young's inequality, it follows that

$$\frac{\kappa_l\,\gamma}{2}\int_\Omega \frac{\kappa(\theta)\,|\nabla(\theta-\tilde{\theta}_0)|^2}{(1+|\theta-\tilde{\theta}_0|)^{1+\gamma}} \leq \frac{\gamma\,\kappa_u}{2\,\kappa_l}\,\|\nabla\tilde{\theta}_0\|^2_{L^2(\Omega)} + \rho_1\,c_V\,L\,\|\nabla\tilde{\theta}_0\|_{L^2(\Omega_1)}\,\|\nabla v\|_{[L^2(\Omega_1)]^9}$$

$$+ \int_\Omega r(\theta)\,|\operatorname{curl} H|^2 + \int_{\Omega_1} \eta(\theta)\,D(v\,,\,v)\,.$$

Making use of (1), we obtain for $\gamma \in]0\,,\,1[$ arbitrary

$$\gamma \int_\Omega \frac{\kappa(\theta)\, |\nabla(\theta - \tilde{\theta}_0)|^2}{(1 + |\theta - \tilde{\theta}_0|)^{1+\gamma}} \leq C_1\,,$$

where the constant C_1 depends on the data trough the previous estimate (1). By the arguments of Lemma A.2.1, we obtain that

$$\|\theta - \tilde{\theta}_0\|_{W^{1,r}_\Gamma(\Omega)} \leq P_r(\|f(\theta)\|_{[L^2(\Omega_1)]^3},\, \|j_0\|_{[L^2(\tilde{\Omega}_c)]^3},\, \|v_g\|_{D^{1,2}(\Omega_1)},\, \|\nabla \tilde{\theta}_0\|_{L^2(\Omega)})\,.$$

for all $1 \leq r < \frac{3}{2}$.

In order to derive the complete estimate (2), we now want to estimate θ on the boundary Σ. We define $\bar{k}_0 := \operatorname*{ess\,sup}_\Gamma \theta_g$. and we recall the definition (6.25) of the numbers M_δ.

Observe that in the case that $M_\delta \leq \bar{k}_0^4$, the estimate

$$\|\theta^4 - M_\delta\|_{L^1(\Sigma)} \leq (\text{meas}(\Sigma) + 1)\, M_\delta \leq 2\, \bar{k}_0^4\, \text{meas}(\Sigma)\,, \tag{6.27}$$

is valid. Suppose now that $M_\delta > \bar{k}_0^4$. For $\gamma > 0$, we introduce the function

$$g_\gamma(t) := \frac{1}{\gamma}\, \text{sign}\big(t\big)\, \min\Big\{\big|t\big|\,,\,\gamma\Big\} + 1\,, \qquad \text{for } t \in \mathbb{R}\,.$$

In (6.12) we choose the test function

$$\xi = \xi_{\delta,\gamma} := g_\gamma(\theta - M_\delta) = \frac{1}{\gamma}\, \text{sign}\big(\theta^4 - M_\delta\big)\, \min\Big\{\big|\theta^4 - M_\delta\big|\,,\,\gamma\Big\} + 1\,.$$

Note that for all $0 < \gamma < M_\delta - \bar{k}_0^4$, the function ξ vanishes on Γ, and observe that $0 \leq \xi \leq 2$ in Ω. On the other hand, since

$$\nabla \xi = \frac{4}{\gamma}\, |\theta|^3\, \chi_{\{|\theta^4 - M_\delta| < \gamma\}}\, \nabla \theta\,,$$

we can verify that

$$|\nabla \xi|^2 \leq \left(\frac{4}{\gamma}\right)^2 (M_\delta + \gamma)^{\frac{3}{2}}\, |\nabla \theta|^2 \in L^1(\Omega)\,,$$

so that we can test with this function in (6.12). Since g_γ is non decreasing, we have $\nabla\theta \cdot \nabla g_\gamma(\theta) = g'_\gamma(\theta)\, |\nabla\theta|^2 \geq 0$, and we obtain that

$$\begin{aligned} \int_\Sigma G(\sigma\, |\theta|^4)\, g_\gamma(\theta) \leq &- \int_{\Omega_1} \rho_1\, c_V\, |v \cdot \nabla \tilde{\theta}_0|\, g_\gamma(\theta) \\ &+ \int_\Omega \Big[r(\theta)\, |\operatorname{curl} H|^2 + \chi_{\Omega_1}\, \eta(\theta)\, D(v\,,\,v)\Big]_{(\delta)}\, g_\gamma(\theta)\,. \end{aligned} \tag{6.28}$$

Now, since Ω is an enclosure and $G(1) \equiv 0$, we can write

$$\begin{aligned} &\int_\Sigma G(\sigma\, |\theta|^4)\, \left[\frac{1}{\gamma}\, \text{sign}(\theta^4 - M_\delta)\, \min\{|\theta^4 - M_\delta|,\, \gamma\} + 1\right] \\ &\quad = \int_\Sigma G\Big(\sigma\, \big[|\theta|^4 - M_\delta\big]\Big)\, \frac{1}{\gamma}\, \text{sign}(\theta^4 - M_\delta)\, \min\{|\theta^4 - M_\delta|,\, \gamma\}\,. \end{aligned}$$

Letting $\gamma \to 0$ in (6.28) it follows that

$$\int_\Sigma G\Big(\sigma\,\big[|\theta|^4 - M_\delta\big]\Big)\,\mathrm{sign}(\theta^4 - M_\delta)$$
$$\leq 2\,\left(\int_{\Omega_1} \rho_1\, c_V\,|v\cdot\nabla\tilde{\theta}_0| + \int_\Omega \Big[r(\theta)\,|\,\mathrm{curl}\,H|^2 + \chi_{\Omega_1}\,\eta(\theta)\,D(v,v)\Big]\right)\,.$$

By the previous estimates and Lemma 4.3.4, we get

$$\|\theta^4 - M_\delta\|_{L^1(\Sigma)} \leq c\left(\|f(\theta)\|^2_{[L^2(\Omega_1)]^3} + \|j_0\|^2_{[L^2(\tilde{\Omega}_c)]^3} + \|v_g\|^2_{D^{1,2}(\Omega_1)} + \|\nabla\tilde{\theta}_0\|^2_{L^2(\Omega)}\right). \quad (6.29)$$

Putting together (6.27) and (6.29), we obtain for all $\delta > 0$ that

$$\|\theta^4 - M_\delta\|_{L^1(\Sigma)} \leq c\left(\|f(\theta)\|^2_{[L^2(\Omega_1)]^3} + \|j_0\|^2_{[L^2(\tilde{\Omega}_c)]^3} + \|v_g\|^2_{D^{1,2}(\Omega_1)} + \|\nabla\tilde{\theta}_0\|^2_{L^2(\Omega)}\right)$$
$$+ 2\,\bar{k}_0^4\,\mathrm{meas}(\Sigma)\,,$$

which finally proves (2). □

Proposition 6.1.4. Let $\big\{v_\delta,\,H_\delta,\,\theta_\delta\big\}$ be any sequence of approximate solutions according to Proposition 6.1.2.

Then there exists $\{v,\,H,\,\theta\} \in D^{1,2}(\Omega) \times \mathcal{H}_\mu(\tilde{\Omega}) \times V^{r,4}(\Omega)$ $(1 \leq r < 3/2)$ and a subsequence $\delta \to 0$ such that

$$v_\delta \longrightarrow v \text{ in } D^{1,2}(\Omega_1)\,,\quad H_\delta \longrightarrow H \text{ in } \mathcal{H}_\mu(\tilde{\Omega})\,,$$
$$\theta_\delta \rightharpoonup \theta \text{ in } W^{1,r}(\Omega)\,,\quad \theta_\delta^4 \longrightarrow \theta^4 \text{ in } L^1(\Sigma)\,.$$

Proof. By the estimates of Proposition 6.1.3, we first find a sequence

$$v_\delta \rightharpoonup v \text{ in } D^{1,2}(\Omega_1)\,,\qquad H_\delta \rightharpoonup H \text{ in } \mathcal{H}_\mu(\tilde{\Omega})\,,\qquad \theta_\delta \rightharpoonup \theta \text{ in } W^{1,r}(\Omega)\,. \quad (6.30)$$

By well-known compactness properties, we now find a (not relabelled) subsequence

$$v_\delta \longrightarrow v \text{ in } L^4(\Omega_1)\,,\quad H_\delta \longrightarrow H \text{ in } L^2(\tilde{\Omega})\,,\quad \theta_\delta \longrightarrow \theta \text{ in } L^r(\Omega)$$
$$\theta_\delta \longrightarrow \theta \text{ in } L^r(\Sigma)\,,\quad \theta_\delta \longrightarrow \theta \text{ almost everywhere in } \Omega\,. \quad (6.31)$$

Passing to the limit $\delta \to 0$ in (6.10), (6.11) by the same arguments as in the proof of Propsition 6.1.2, we see that the pair $\{v,\,H\}$ satisfies the relations

$$\int_{\Omega_1} \rho_1\,(v\cdot\nabla)v\cdot\phi + \int_{\Omega_1} \eta(\theta)\,D(v\,,\,\phi) = \int_{\Omega_1} \big(\mathrm{curl}\,H \times \mu\,H\big)\cdot\phi + \int_{\Omega_1} f(\theta)\cdot\phi\,,$$
$$\int_{\tilde{\Omega}} r(\theta)\;\mathrm{curl}\,H\cdot\mathrm{curl}\,\psi = \int_{\tilde{\Omega}} \big(v \times \mu\,H\big)\cdot\mathrm{curl}\,\psi\,.$$

In these relations, we now use the test-functions $\phi = v_\delta - v\,,\ \psi = H_\delta - H$. We do the same in the identities (6.10) and (6.11). Subtracting the two arising integral relations, we can write

$$\int_{\Omega_1} \eta(\theta_\delta)\, D(v_\delta - v\,,\, v_\delta - v) = -\int_{\Omega_1} [\eta(\theta_\delta) - \eta(\theta)]\, D(v\,,\, v_\delta - v)$$
$$- \int_{\Omega_1} \rho_1\ (v_\delta \cdot \nabla v_\delta - v \cdot \nabla v) \cdot (v_\delta - v)$$
$$- \int_{\Omega_1} \Big(\big(\operatorname{curl} H_\delta \times \mu\, H_\delta\big) - \big(\operatorname{curl} H \times \mu\, H\big)\Big) \cdot (v_\delta - v) - \int_{\Omega_1} [f(\theta_\delta) - f(\theta)] \cdot (v_\delta - v)\,,$$

as well as

$$\int_{\tilde\Omega} r(\theta_\delta)\, |\operatorname{curl}(H_\delta - H)|^2 = -\int_{\tilde\Omega} [r(\theta_\delta) - r(\theta)]\ \operatorname{curl} H \cdot \operatorname{curl}(H_\delta - H)$$
$$+ \int_{\Omega_1} \Big(\big(v_\delta \times \mu\, H_\delta\big) - \big(v \times \mu\, H\big)\Big) \cdot \operatorname{curl}(H_\delta - H)\,.$$

Now, by (6.30) and (6.31), it is not difficult to see that the right-hand sides of both relations converge to zero for $\delta \to 0$, proving that

$$v_\delta \longrightarrow v \text{ in } D^{1,2}(\Omega_1)\,, \qquad H_\delta \longrightarrow H \text{ in } \mathcal{H}_\mu(\tilde\Omega)\,.$$

Thus, we have also

$$D(v_\delta\,,\, v_\delta) \longrightarrow D(v\,,\, v) \text{ in } L^1(\Omega_1)\,, \qquad |\operatorname{curl} H_\delta|^2 \longrightarrow |\operatorname{curl} H|^2 \text{ in } L^1(\tilde\Omega)\,,$$

which yields

$$\Big[r(\theta_\delta)\, |\operatorname{curl} H_\delta|^2 + \chi_{\Omega_1}\, \eta(\theta_\delta)\, D(v_\delta\,,\, v_\delta)\Big]_{(\delta)} \longrightarrow r(\theta)\, |\operatorname{curl} H|^2 + \chi_{\Omega_1}\, \eta(\theta)\, D(v\,,\, v)\,, \tag{6.32}$$

strongly in $L^1(\Omega)$. Now we prove the convergence property for the boundary integral.

First, we prove that the sequence of numbers M_δ given by (6.25) is bounded. Using estimate Proposition 6.1.3, (2) and Fatou's lemma, we can write that

$$\int_\Sigma \liminf_{\delta\to 0} |\theta_\delta^4 - M_\delta| \ \le \liminf_{\delta\to 0} \int_\Sigma |\theta_\delta^4 - M_\delta| \le C\,. \tag{6.33}$$

Seeking a contradiction, we suppose that there exists a subsequence $M_\delta \to \infty$. For this subsequence, we obtain that

$$\liminf_{\delta\to 0} |\theta_\delta^4 - M_\delta| = \lim_{\delta\to 0} |\theta_\delta^4 - M_\delta| = \lim_{\delta\to 0} |\theta^4 - M_\delta| = +\infty \qquad \text{almost everywhere on } \Sigma\,,$$

since the pointwise limes θ^4 is almost everywhere finite. This contradicts (6.33).

Thus, the sequence $\{M_\delta\}$ is bounded, which by definition also implies a uniform bound $\| \theta_\delta^4 \|_{L^1(\Sigma)} \le C$.

The assumptions of Proposition 4.4.1 are satisfied, which proves the result. □

We are now able to prove the main result of this section.

Proof of Theorem 6.1.1. Thanks to the convergence properties stated by Prop. 6.1.4, we find a triple

$$\{v,\, H,\, \theta\} \in D^{1,2}(\Omega_1) \times \mathcal{H}_\mu(\tilde{\Omega}) \times V^{r,4}(\Omega), \quad 1 \le r < \frac{3}{2} \text{ arbitrary}\,,$$

such that $v = v_g$ on $\partial\Omega_1$, $\theta = \theta_g$ on Γ, $\operatorname{curl} H = j_0$ in $\tilde{\Omega}_{c_0}$, and the relations

$$\begin{aligned}
&\int_{\Omega_1} \rho_1\,(v\cdot\nabla)v\cdot\phi + \int_{\Omega_1} \eta(\theta)\, D(v\,,\,\phi) = \int_{\Omega_1} \big(\operatorname{curl} H \times \mu\, H\big)\cdot\phi + \int_{\Omega_1} f(\theta)\cdot\phi\,,\\
&\int_{\tilde{\Omega}} r(\theta)\, \operatorname{curl} H\cdot\operatorname{curl}\psi = \int_{\Omega_1} \big(v\times\mu\, H\big)\cdot\,\operatorname{curl}\psi\,,\\
&\int_{\Omega_1} \rho_1\, c_V\, v\cdot\nabla\theta\,\xi + \int_{\Omega} \kappa(\theta)\,\nabla\theta\cdot\,\nabla\xi + \int_{\Sigma} G(\sigma\,\theta^4)\,\xi\\
&\qquad\qquad = \int_{\Omega} \Big(r(\theta)\,|\operatorname{curl} H|^2 + \eta(\theta)\,\chi_{\Omega_1}\, D(v\,,\,v)\Big)\,\xi\,,
\end{aligned} \tag{6.34}$$

are satisfied for all $\{\phi,\,\psi,\,\xi\} \in D^{1,2}_0(\Omega_1) \times \mathcal{H}^0_\mu(\tilde{\Omega}) \times V^{r',\infty}_\Gamma(\Omega)$. By the result (6.14) and Sobolev's embedding theorems, we find that $\operatorname{curl} H \times \mu\, H \in L^{6/5}(\tilde{\Omega})$. This proves the claim. □

6.2 Boussinesq approximation

In the first section, we replaced the Boussinesq approximation of the gravitational force (3.12) by the *bounded* term (6.1). We can argue in favor of this choice by observing that the Boussinesq approximation is valid only in the range of *small* density variations, that is

$$0 \le \alpha\,(\theta - \theta_M) \ll 1\,. \tag{6.35}$$

This approach would be fully justify if we could prove *a posteriori* that the weak solutions obtained in the first section actually satisfy (6.35). We cannot give a proof of such a full justification. Instead, we have a weaker result.

Lemma 6.2.1. Assume that the hypothesis of Theorem 6.1.1 are satisfied, and assume in addition that θ_g is a constant. Let the numbers α, M_θ in (6.1) be such that

$$1 - \bar{c}\,\frac{\operatorname{meas}(\Omega_1)\,\rho_1^2\,|\vec{g}|^2}{\kappa_l}\, M_\theta\,\alpha > 0\,,$$

where $\bar{c} = \sqrt{2}\,c\,c_0^2$, with the constant c that appears in Proposition 6.1.3, (1) and the constant c_0 of the embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$.

Then, for the weak solution of (P) constructed in Theorem 6.1.1, the estimate

$$\left(\frac{1}{\operatorname{meas}(\Omega_1)} \int_{\Omega_1} \alpha^2 \, |\theta - \theta_M|^2\right)^{1/2} \leq \frac{\bar{c}\, \alpha \, (\|j_0\|^2_{[L^2(\tilde{\Omega}_{c_0})]^3} + \|v_g\|^2_{D^{1,2}(\Omega_1)})}{\kappa_l - \bar{c} \operatorname{meas}(\Omega_1)\, \rho_1^2 \, |\vec{g}|^2 \, \alpha \, M_\theta} \,,$$

is valid.

Proof. We consider the approximate solutions $\{v_\delta,\, H_\delta,\, \theta_\delta\}$ according to Proposition 6.1.2 and derive an additional uniform estimate. We start from (6.12), and we write v, H, θ instead of v_δ, H_δ, θ_δ.

For a parameter $\lambda > 0$ we are allowed to use the test function

$$\xi = (\theta - \theta_g)^{(\lambda)} = \operatorname{sign}(\theta - \theta_g) \, \min\{|\theta - \theta_g|,\, \lambda\} \,.$$

Since we assume that θ_g is constant, it immediately follows that

$$\begin{aligned} &\int_\Omega \kappa(\theta) \, |\nabla(\theta - \theta_g)^{(\lambda)}|^2 + \int_\Sigma G(\sigma \, \theta^4) \, (\theta - \theta_g)^{(\lambda)} \\ &\qquad = \int_\Omega \Big[\chi_{\Omega_1} \, \eta(\theta) \, D(v,v) + r(\theta) \, |\operatorname{curl} H|^2\Big]_{(\delta)} (\theta - \theta_g)^{(\lambda)} \,. \end{aligned} \tag{6.36}$$

Using the selfadjointness of the operator G and the fact that $G(1) \equiv 0$ on Σ, we can write

$$\int_\Sigma G(\sigma \, \theta^4) \, (\theta - \theta_g)^{(\lambda)} = \int_\Sigma G(\sigma \, \theta^4) \, [(\theta - \theta_g)^{(\lambda)} + \min\{\theta_g,\, \lambda\}] \,.$$

We see that the function

$$F(s) := [(s - \theta_g)^{(\lambda)} + \min\{\theta_g,\, \lambda\}] \qquad \text{for } s \in \mathbb{R} \,,$$

satisfies the assumptions of Lemma 4.3.6. Therefore, (6.36) leads to the inequality

$$\int_\Omega \kappa(\theta) \, |\nabla(\theta - \theta_g)^{(\lambda)}|^2 \leq \int_\Omega \Big[\chi_{\Omega_1} \, \eta(\theta) \, D(v,v) + r(\theta) \, |\operatorname{curl} H|^2\Big]_{(\delta)} (\theta - \theta_g)^{(\lambda)} \,.$$

Using (1), we find that

$$\begin{aligned} \int_\Omega \kappa(\theta) \, |\nabla(\theta - \theta_g)^{(\lambda)}|^2 &\leq \lambda \, (\int_{\Omega_1} \eta(\theta) \, D(v,v) + \int_\Omega r(\theta) \, |\operatorname{curl} H|^2) \\ &\leq c \, (\|f(\theta)\|^2_{[L^2(\Omega_1)]^3} + \| \, j_0 \, \|^2_{[L^2(\tilde{\Omega}_{c_0})]^3} + \|v_g\|^2_{D^{1,2}(\Omega_1)}) \, \lambda \,. \end{aligned} \tag{6.37}$$

On the other hand, using the continuity of the embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, we find that

$$\int_\Omega \kappa(\theta) \, |\nabla(\theta - \theta_g)^{(\lambda)}|^2 \geq c_0^{-2} \, \kappa_l \, \|(\theta - \theta_g)^{(\lambda)}\|^2_{L^6(\Omega)} \,.$$

This together with (6.37) obviously gives that

$$c_0^{-2}\,\kappa_l\,\lambda^2 \text{ meas}\Big(\{x\in\Omega \,:\, |\theta-\theta_g|\geq\lambda\}\Big)^{1/3}$$
$$\leq c\,(\|f(\theta)\|^2_{[L^2(\Omega_1)]^3}+\|\,j_0\,\|^2_{[L^2(\tilde\Omega_{c_0})]^3}+\|v_g\|^2_{D^{1,2}(\Omega_1)})\,\lambda\,.$$

Therefore,

$$\sup_{\lambda>0}\left\{\lambda \text{ meas}\Big(\{x\in\Omega_1 \,:\, |\theta-\theta_g|\geq\lambda\}\Big)^{1/3}\right\}$$
$$\leq \frac{c\,c_0^2}{\kappa_l}\,(\|f(\theta_\delta)\|^2_{[L^2(\Omega_1)]^3}+\|\,j_0\,\|^2_{[L^2(\tilde\Omega_{c_0})]^3}+\|v_g\|^2_{D^{1,2}(\Omega_1)})\,.$$

Now, we apply the embedding properties of the weak L^p-spaces (see Lemma A.2.2) in order to obtain that

$$\|\theta-\theta_g\|_{L^2(\Omega_1)}\leq\sqrt{2}\text{ meas}(\Omega_1)^{1/2}\sup_{\lambda>0}\left\{\lambda \text{ meas}\Big(\{x\in\Omega_1 \,:\, |\theta-\theta_g|\geq\lambda\}\Big)^{1/3}\right\}$$
$$\leq\frac{\bar c\text{ meas}(\Omega_1)^{1/2}}{\kappa_l}\,(\|f(\theta)\|^2_{[L^2(\Omega_1)]^3}+\|\,j_0\,\|^2_{[L^2(\tilde\Omega_{c_0})]^3}+\|v_g\|^2_{D^{1,2}(\Omega_1)})\,. \quad (6.38)$$

On the other hand, we use the estimate (6.2), and can write

$$\|f(\theta)\|^2_{[L^2(\Omega_1)]^3}\leq\rho_1^2\,|\vec g|^2\,M_\theta\int_{\Omega_1}\alpha\,|\theta-\theta_M|\leq\rho_1^2\,|\vec g|^2\,M_\theta\,\alpha\text{ meas}(\Omega_1)^{1/2}\,\|\theta-\theta_M\|_{L^2(\Omega_1)}\,. \quad (6.39)$$

In view of (6.38), we then have

$$\left(1-\frac{\bar c\text{ meas}(\Omega_1)\,\rho_1^2\,|\vec g|^2}{\kappa_l}\,M_\theta\,\alpha\right)\,\|\theta-\theta_M\|_{L^2(\Omega_1)}$$
$$\leq\frac{\bar c\text{ meas}(\Omega_1)^{1/2}}{\kappa_l}\,(\|\,j_0\,\|^2_{[L^2(\tilde\Omega_{c_0})]^3}+\|v_g\|^2_{D^{1,2}(\Omega_1)})\,.$$

We recall that $\theta=\theta_\delta$. The claim follows, since the last estimate is preserved in the limit $\delta\to 0$. □

Assuming that the hypothesis of Theorem 6.1.1 are satisfied, we prove the following result.

Theorem 6.2.2. Let the assumptions of Theorem 6.1.1 be satisfied, but let f be given by (3.12). If the coefficient α is sufficiently small with respect to the other data, the existence result of Theorem 6.1.1 holds true.

The rest of the section is devoted to the proof of this theorem. We will use the same notations as in Propostion 6.1.2. We additionally introduce the notation

$$J_n(\Omega_1)\,:=\,\left\{u\in[L^2(\Omega_1)]^3\,\middle|\,\text{div}\,u=0\ \text{ in }\Omega_1\,,\ u\cdot\vec n=0\ \text{ on }\partial\Omega_1\right\},$$

where the constraints are intended in the sense of the generalized *div* operator.

Proposition 6.2.3. Let $\delta > 0$ be an arbitrary number. Suppose that the assumptions of Theorem 6.2.2 are satisfied. If $\{\tilde{v},\, \tilde{H},\, \tilde{\theta}\}$ is an arbitrary element of $J_n(\Omega_1) \times [L^2(\tilde{\Omega})]^3 \times L^2(\Omega)$, then there exists a unique triple

$$\{v,\, H,\, \theta\} \in D^{1,2}(\Omega_1) \times \mathcal{H}_\mu(\tilde{\Omega}) \times V^{2,5}(\Omega)\,,$$

such that $v = v_g$ on $\partial\Omega_1$, $\theta = \theta_g$ on Γ, $\operatorname{curl} H = j_0$ in $\tilde{\Omega}_{c_0}$, and

$$\int_{\Omega_1} \rho_1\, (\tilde{v}\cdot\nabla)v\cdot\phi + \int_{\Omega_1} \eta(\tilde{\theta})\, D(v\,,\,\phi) = \int_{\Omega_1} \big(\operatorname{curl} H \times [\mu\,\tilde{H}]_{(\delta)}\big)\cdot\phi + \int_{\Omega_1} f(\tilde{\theta})\cdot\phi\,, \tag{6.40}$$

$$\int_{\tilde{\Omega}} r(\tilde{\theta})\, \operatorname{curl} H\cdot\operatorname{curl}\psi = \int_{\Omega_1} \big(v\times[\mu\,\tilde{H}]_{(\delta)}\big)\cdot\operatorname{curl}\psi\,, \tag{6.41}$$

$$\begin{aligned}\int_{\Omega_1} \rho\, c_V\, \tilde{v}\cdot\nabla\theta\,\xi + \int_{\Omega} \kappa(\tilde{\theta})\,\nabla\theta\cdot\nabla\xi + \int_{\Sigma} G(\sigma\,\theta^4)\,\xi \\ = \int_{\Omega} \Big[r(\tilde{\theta})\,|\operatorname{curl} H|^2 + \chi_{\Omega_1}\,\eta(\tilde{\theta})\,D(v\,,\,v)\Big]_{(\delta)}\,\xi\,.\end{aligned} \tag{6.42}$$

are satisfied for all $\{\phi,\, \psi,\, \xi\} \in D_0^{1,2}(\Omega_1) \times \mathcal{H}^0_\mu(\tilde{\Omega}) \times V^{2,5}_\Gamma(\Omega)$. In addition

$$\theta \ge \operatorname*{ess\,inf}_{\Gamma} \theta_g \qquad \text{almost everywhere in } \Omega\,.$$

Proof. Existence is a routine matter and is proved, for example, by the method of Proposition 6.1.2.

We prove the uniqueness. Suppose that both $\{v_1,\, H_1,\, \theta_1\}$ and $\{v_2,\, H_2,\, \theta_2\}$ satisfy the integral relations (6.40), (6.41) and (6.42). Then, in (6.40) written alternatively for v_1 and v_2, we test with $v_1 - v_2$ and subtract both results. We do the same in (6.41). We observe that

$$\int_{\Omega_1} \rho_1\, \Big(\tilde{v}\cdot\nabla(v_1 - v_2)\Big)\cdot(v_1 - v_2) = 0\,.$$

We obtain the two relations

$$\int_{\Omega_1} \eta(\tilde{\theta})\, D(v_1 - v_2\,,\, v_1 - v_2) = \int_{\Omega_1} \big(\operatorname{curl}(H_1 - H_2)\times[\mu\,\tilde{H}]_{(\delta)}\big)\cdot(v_1 - v_2)\,,$$

$$\int_{\tilde{\Omega}} r(\tilde{\theta})\,|\operatorname{curl}(H_1 - H_2)|^2 = \int_{\Omega_1} \big((v_1 - v_2)\times[\mu\,\tilde{H}]_{(\delta)}\big)\cdot\operatorname{curl}(H_1 - H_2)\,,$$

which clearly imply, after addition, that $v_1 = v_2$ and $H_1 = H_2$. Now, for $\gamma > 0$, we use in (6.42) the test function $g_\gamma := \min\{(\theta_1 - \theta_2)^+\,,\,\gamma\}$, and observing that $\int_{\Omega_1} \rho_1\, c_V\, \tilde{v}\cdot\nabla(\theta_1 - \theta_2)\, g_\gamma = 0$, we obtain the relation

$$\int_{\Omega} \kappa(\tilde{\theta})\,\nabla(\theta_1 - \theta_2)\cdot\nabla g_\gamma + \int_{\Sigma} G\Big(\sigma\,\big[\theta_1^4 - \theta_2^4\big]\Big)\, g_\gamma = 0\,.$$

By the arguments of Laitinen and Tiihonen [2001] (see also Druet [2009a]), this leads to the uniqueness. $\square$

Proposition 6.2.3 provides us with a well-defined, obviously compact mapping

$$T_\delta : \, J_n(\Omega_1) \times [L^2(\tilde\Omega)]^3 \times L^2(\Omega) \longrightarrow J_n(\Omega_1) \times [L^2(\tilde\Omega)]^3 \times L^2(\Omega)$$
$$\{\tilde v, \, \tilde H, \, \tilde\theta\} \longmapsto \{v, \, H, \, \theta\} \, . \tag{6.43}$$

Next we show the

Lemma 6.2.4. If the coefficient α is sufficiently small with respect to the other data, the mapping T_δ given by (6.43) satisfies the assumptions of the Schauder fixed-point principle. (In the simplified case of constant coefficients and boundary data, the smallness assumption on α is formulated more precisely in the equation (6.47) below.)

Proof. To prove the continuity of T_δ is, again, a routine matter. We have to consider an arbitrary sequence $\{\tilde v_k, \, \tilde H_k, \, \tilde\theta_k\}$ in $J_n(\Omega_1) \times [L^2(\tilde\Omega)]^3 \times L^2(\Omega)$ such that

$$\{\tilde v_k, \, \tilde H_k, \, \tilde\theta_k\} \longrightarrow \{\tilde v, \, \tilde H, \, \tilde\theta\} \;\text{ in } J_n(\Omega_1) \times [L^2(\tilde\Omega)]^3 \times L^2(\Omega) \, .$$

Choosing an arbitrary subsequence, that we not relabel, we will find by the compactness properties of T_δ a sub-subsequence such that $T_\delta\Big(\{\tilde v_k, \, \tilde H_k, \, \tilde\theta_k\}\Big) \longrightarrow w$ in $J_n(\Omega_1) \times [L^2(\tilde\Omega)]^3 \times L^2(\Omega)$. By arguments similar to the proof of Proposition 6.1.2, that we do not want to repeat in detail, and the uniqueness obtained in Proposition 6.2.3, we show that $w = T_\delta\Big(\{\tilde v, \, \tilde H, \, \tilde\theta\}\Big)$. Then, strong convergence follows for the entire sequence.

We finally prove that T_δ maps some closed convex set into itself. In order to easier arrive at an estimate, we prove the claim in the simplified case that $v_g = 0$, that θ_g is constant, and, all coefficients are piecewise constants. At the expense of technical complications, one verifies that the result is qualitatively preserved in the general case. Inserting v in (6.40) and H in (6.41), we obtain the estimate (c.p. Proposition 6.1.3, (1))

$$\int_{\Omega_1} \eta \, D(v \, , \, v) + \int_{\tilde\Omega} r \, | \operatorname{curl} H|^2 \le \frac{L^2}{\eta} \, \|f(\tilde\theta)\|^2_{[L^2(\Omega_1)]^3} + \int_{\tilde\Omega} r \, |j_0|^2 \, . \tag{6.44}$$

Arguing now as in the proof of Lemma 6.2.1, we verify that the solution $T_\delta\{\tilde v, \, \tilde H, \, \tilde\theta\}$ satisfies

$$\|\theta - \theta_g\|_{L^2(\Omega_1)} \le \frac{\bar c \; \text{meas}(\Omega_1)^{1/2}}{\kappa_l} \, (\|f(\tilde\theta)\|^2_{[L^2(\Omega_1)]^3} + \| \; j_0 \; \|^2_{[L^2(\tilde\Omega_{c_0})]^3} + \|v_g\|^2_{D^{1,2}(\Omega_1)}) \, . \tag{6.45}$$

To estimate $\|f(\tilde\theta)\|_{[L^2(\Omega_1)]^3}$ as in (6.39) is not possible anymore. Instead, we simply assume that $\tilde\theta - \tilde\theta_M \in \overline{B_X(0)} (\subset L^2(\Omega_1))$ for some $X > 0$, and we obtain that

$$\|\theta - \theta_M\|_{L^2(\Omega_1)} \le \frac{\bar c \; \text{meas}(\Omega_1)^{1/2}}{\kappa_l} \, (\rho_1^2 \, |\vec g|^2 \, \alpha^2 \, X^2 + \| \; j_0 \; \|^2_{[L^2(\tilde\Omega_{c_0})]^3} + \|v_g\|^2_{D^{1,2}(\Omega_1)}) \, . \tag{6.46}$$

We introduce

$$a_1 := \frac{\bar c \; \text{meas}(\Omega_1)^{1/2}}{\kappa_l} \, \rho_1^2 \, |\vec g|^2 \, \alpha^2, \quad a_0 := \frac{\bar c \; \text{meas}(\Omega_1)^{1/2}}{\kappa_l} \, (\| \; j_0 \; \|^2_{[L^2(\tilde\Omega_{c_0})]^3} + \|v_g\|^2_{D^{1,2}(\Omega_1)}) \, .$$

Under the condition,

$$1-4\,\frac{\bar{c}^2\ \mathrm{meas}(\Omega_1)}{\kappa_l^2}\,\rho_1^2\,|\vec{g}|^2\,\alpha^2\,(\|\ j_0\ \|^2_{[L^2(\tilde{\Omega}_{c_0})]^3}+\|v_g\|^2_{D^{1,2}(\Omega_1)})>0\,, \tag{6.47}$$

we see that the equation $X=a_1\,X^2+a_0$ has the positive solution

$$X=\frac{2\,a_0}{1+\sqrt{1-4\,a_0\,a_1}}\le 2\,\frac{\bar{c}\ \mathrm{meas}(\Omega_1)^{1/2}}{\kappa_l}\,(\|\ j_0\ \|^2_{[L^2(\tilde{\Omega}_{c_0})]^3}+\|v_g\|^2_{D^{1,2}(\Omega_1)})\,. \tag{6.48}$$

We then define a closed convex set $M=M(X)\subset L^2(\Omega)$ by

$$M:=\left\{\tilde{\theta}\in L^2(\Omega)\,\middle|\,\tilde{\theta}-\tilde{\theta}_M\in\overline{B_X(0)}(\subset L^2(\Omega_1))\right\}.$$

Note in view of (6.46) that $\tilde{\theta}\in M$ implies $\theta\in M$. In view of (6.44), we then easily find numbers Y_1, Y_2, Y_3 depending on X and on the data such that T_δ maps the set

$$\overline{B_{Y_1}(0)}\times\overline{B_{Y_2}(0)}\times M\cap\overline{B_{Y_3}(0)}\ \subset\ J_n(\Omega_1)\times[L^2(\tilde{\Omega})]^3\times L^2(\Omega)\,,$$

into itself. □

Now, we prove the main result of this section.

Proof of Theorem 6.2.2. By Proposition 6.2.3 and Lemma 6.2.4, the Schauder fixed-point theorem gives the existence of a triple $\{v_\delta,\,H_\delta,\,\theta_\delta\}\in D^{1,2}(\Omega_1)\times\mathcal{H}_\mu(\tilde{\Omega})\times V^{2,5}(\Omega)$ such that $v=v_g$ on $\partial\Omega_1$, $\theta=\theta_g$ on Γ, $\operatorname{curl}H=j_0$ in $\tilde{\Omega}_{c_0}$, and

$$\begin{aligned}
&\int_{\Omega_1}\rho_1\,(v_\delta\cdot\nabla)v_\delta\cdot\phi+\int_{\Omega_1}\eta(\theta_\delta)\,D(v_\delta\,,\,\phi)=\int_{\Omega_1}\big(\mathrm{curl}H_\delta\times[\mu\,H_\delta]_{(\delta)}\big)\cdot\phi+\int_{\Omega_1}f(\theta_\delta)\cdot\phi\,,\\
&\int_{\tilde{\Omega}}r(\theta_\delta)\ \operatorname{curl}H_\delta\cdot\operatorname{curl}\psi=\int_{\Omega_1}\big(v_\delta\times[\mu\,H_\delta]_{(\delta)}\big)\cdot\ \operatorname{curl}\psi\,,\\
&\int_{\Omega_1}\rho\,c_V\,v_\delta\cdot\nabla\theta_\delta\,\xi+\int_\Omega\kappa(\theta_\delta)\,\nabla\theta_\delta\cdot\ \nabla\xi+\int_\Sigma G(\sigma\,|\theta_\delta|^4)\,\xi\\
&\qquad\qquad=\int_\Omega\Big[r(\theta_\delta)\,|\operatorname{curl}H_\delta|^2+\chi_{\Omega_1}\,\eta(\theta_\delta)\,D(v_\delta\,,\,v_\delta)\Big]_{(\delta)}\,\xi\,.
\end{aligned}$$

We pass to the limit with the same strategy as in the first section. In order to obtain the strong convergence

$$v_\delta\longrightarrow v\ \text{ in }D^{1,2}(\Omega_1)\,,\quad H_\delta\longrightarrow H\ \text{ in }\mathcal{H}_\mu(\tilde{\Omega})\,,$$

the particular form $f(\theta_\delta)\ =\ -\,\rho_1\,\vec{g}\,\alpha\,(\theta_\delta-\theta_{M,\delta})$ means no particular difficulty. In the limit, the relations

$$\int_{\Omega_1} \rho_1\,(v\cdot\nabla)v\cdot\phi + \int_{\Omega_1} \eta(\theta)\,D(v\,,\,\phi) = \int_{\Omega_1} \big(\mathrm{curl}H\times\mu\,H\big)\cdot\phi + \int_{\Omega_1} f(\theta)\cdot\phi\,,$$

$$\int_{\tilde\Omega} r(\theta)\;\mathrm{curl}\,H\cdot\mathrm{curl}\,\psi = \int_{\Omega_1} \big(v\times\mu\,H\big)\cdot\,\mathrm{curl}\,\psi\,,$$

$$\int_{\Omega_1} \rho\,c_V\,v\cdot\nabla\theta\,\xi + \int_{\Omega} \kappa(\theta)\,\nabla\theta\cdot\,\nabla\xi + \int_{\Sigma} G(\sigma\,|\theta|^4)\,\xi$$

$$= \int_{\Omega} \Big(r(\theta)\,|\,\mathrm{curl}\,H|^2 + \chi_{\Omega_1}\,\eta(\theta)\,D(v\,,\,v)\Big)\,\xi\,, \tag{6.49}$$

are satisfied, proving the existence of a weak solution. In addition, we can control the L^2-norm of the density fluctuations by a continuous function of the data. In the simplified case that $v_g = 0$ and that θ_g is constant, we obtain in view of (6.48) that

$$\left(\frac{1}{\mathrm{meas}(\Omega_1)}\int_{\Omega_1} \alpha^2\,|\theta-\theta_M|^2\right)^{1/2} \le \frac{2\,\bar c\,\alpha}{\kappa_l}\;\|\,j_0\,\|^2_{[L^2(\tilde\Omega_{c_0})]^3}\;. \tag{6.50}$$

$\square$

6.3 A uniqueness result for small data

Through the section, we assume for simplicity that the temperature-dependent force term f in the Navier-Stokes equations has the form (7.1) and is bounded. We need to make the technical assumptions that

$$\begin{cases} \text{The coefficients } \eta,\ r,\ \kappa \text{ are piecewise constant.} \\ \text{There exists } 1\le\tilde p<2 \text{ such that } \partial\Omega_i \text{ is of class } \mathcal{C}^{1,1/\tilde p} \text{ for } i=0,\dots,m\,. \end{cases} \tag{6.51}$$

We introduce additional notations. We define a Banach space X for the data $\{v_g\,j_0,\,\theta_g\}$ of (P_{st}) by

$$X := D^{1,2}(\Omega_1)\cap L^\infty(\Omega_1)\times[L^2(\tilde\Omega_{c_0})]^3\times W^{1,2}(\Omega)\cap L^\infty(\Omega)\,,$$

and for the weak solutions $\{v,\,H,\,\theta\}$ of (P_{st}) a Banach space Y by

$$Y := D^{1,2}(\Omega_1)\times\mathcal{H}_\mu(\tilde\Omega)\times V^{r,4}(\Omega)\,,$$

for some $r<3/2$. For each given triple of data $\{v_g,\,j_0,\,\theta_g\}\in X$, we can define a solution subset S of the space Y by

$$S\big(\{v_g,\,j_0,\,\theta_g\}\big)$$
$$:= \Big\{\{v,\,H,\,\theta\}\in Y\;\Big|\;\{v,\,H,\,\theta\}\text{ is a weak solution of }(P_{\mathrm{st}})\text{ for the data }\{v_g,\,j_0,\,\theta_g\}\Big\}\,.$$

Further, we set

$$\hat{K}_0 = \hat{K}_0\big(\{v_g,\, j_0,\, \theta_g\}\big) := \|j_0\|_{[L^2(\tilde{\Omega}_{c_0})]^3} + \|v_g\|_{D^{1,2}(\Omega_1)} + \|\nabla\theta_g\|_{[L^2(\Omega)]^3}\,, \tag{6.52}$$

which represents a norm of the data of the problem (P_{st}). For numbers $\theta_{\max} > 0$, we introduce the set

$$X_{\theta_{\max}} := \Big\{\{v_g,\, j_0,\, \theta_g\} \in X \,\Big|\, \|\theta_g\|_{L^\infty(\Omega)} \le \theta_{\max}\Big\}\,,$$

which is convex and closed in X. We denote by c_0 the constant of the continuous embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$.

Theorem 6.3.1. Assume that (6.51) is satisfied, and let $\theta_{\max} > 0$ be arbitrary, but fixed. If the number M_θ in (6.1) satisfies

$$M_\theta < \frac{\eta^2\,\kappa_l}{\bar{c}\,L^2\,(1+\mathrm{diam}(\Omega))^2\ \mathrm{meas}(\Omega_1)\,|\vec{g}|^2\,\rho_1^2\,\alpha}\,, \tag{6.53}$$

with $\bar{c} = \sqrt{2}\,c_{\mathrm{Korn}}\,c_0^2$, then there exists $\epsilon > 0$ such that for all $\{v_g,\, j_0,\, \theta_g\} \in X_{\theta_{\max}}$ that satisfy $\hat{K}_0 \le \epsilon$, the set $S\big(\{v_g,\, j_0,\, \theta_g\}\big) \subset Y$ consists of at most one element.

Remark 6.3.2. Whenever the stationary Navier-Stokes system is involved, the uniqueness of weak solutions can be proved only for *small external forces* (see for example Temam [1977], Ch. II, Paragraphs 1 and 4). In the case of the coupled model presently under study, Theorem 6.3.1 shows that the uniqueness issue is related to two additional parameters: the importance of the *temperature fluctuations* in the fluid, measured by the number M_θ, and the maximal imposed temperature $\theta_{\max}$.

The proof of Theorem 6.3.1 is based on several auxiliary results.

Lemma 6.3.3. Let θ_g be a positive constant. Assume that $\{v,\, H,\, \theta\} \in S\big(\{0,\, 0,\, \theta_g\}\big)$.
If the assumptions of Theorem 6.3.1 are satisfied, then $\{v,\, H\} = 0$ and $\theta \equiv \theta_g$.

Proof. We test the integral relations (3.62) and (3.63) respectively with v and H, and obtain, after addition, the energy equality

$$\int_{\Omega_1} \eta\, D(v,v) + \int_{\tilde{\Omega}} r\,|\,\mathrm{curl}\, H|^2 = \int_{\Omega_1} f(\theta)\cdot v\,.$$

Using standard inequalities, we derive the estimate

$$\int_{\Omega_1} \eta\, D(v,v) + \int_{\tilde{\Omega}} r\,|\,\mathrm{curl}\, H|^2 \le c_{\mathrm{Korn}}\,\frac{L^2}{\eta^2}\,\|f(\theta)\|^2_{[L^2(\Omega_1)]^3}\,. \tag{6.54}$$

For a parameter $\lambda > 0$ and $s \in \mathbb{R}$, we introduce the function $s^{(\lambda)} := \mathrm{sign}(s)\,\min\{|s|, \lambda\}$. Denoting by Ψ a primitive of the function $(\cdot)^{(\lambda)}$, we observe that

$$\int_{\Omega_1} \rho_1\, c_V\, v\cdot\nabla\theta\,(\theta-\theta_g)^{(\lambda)} = \int_{\Omega_1} \rho_1\, c_V\, v\cdot\nabla\Psi(\theta-\theta_g) = 0\,.$$

In view of Lemma 4.3.6, we have also $\int_\Sigma G(\sigma\,\theta^4)\,(\theta-\theta_g)^{(\lambda)} \geq 0$. Using (6.54), we obtain from (3.64) and Lemma 6.3.6 below the inequality

$$\int_\Omega \kappa\,|\nabla(\theta-\theta_g)^{(\lambda)}|^2 \leq \int_\Omega \big(\eta\,D(v,v)+r\,|\operatorname{curl} H|^2\big)\,(\theta-\theta_g)^{(\lambda)} \leq \lambda\, c_{\mathrm{Korn}}\,\frac{L^2}{\eta^2}\,\|f(\theta)\|^2_{[L^2(\Omega_1)]^3}\,. \tag{6.55}$$

Now, we can write

$$\begin{aligned}\int_\Omega \kappa\,|\nabla(\theta-\theta_g)^{(\lambda)}|^2 &\geq \frac{\kappa_l}{(1+\operatorname{diam}(\Omega))^2}\,\|(\theta-\theta_g)^{(\lambda)}\|^2_{W^{1,2}(\Omega)}\\ &\geq \frac{\kappa_l}{c_0^2\,(1+\operatorname{diam}(\Omega))^2}\,\|(\theta-\theta_g)^{(\lambda)}\|^2_{L^6(\Omega)}\\ &\geq \frac{\kappa_l}{c_0^2\,(1+\operatorname{diam}(\Omega))^2}\,\|(\theta-\theta_g)^{(\lambda)}\|^2_{L^6(\Omega_1)}\\ &\geq \frac{\kappa_l}{c_0^2\,(1+\operatorname{diam}(\Omega))^2}\,\lambda^2\ \operatorname{meas}(\{x\in\Omega_1\,:\,|\theta-\theta_g|>\lambda\})^{\frac{1}{3}}\,.\end{aligned}$$

From (6.55), it now follows that for all $\lambda>0$,

$$\frac{\kappa_l}{c_0^2\,(1+\operatorname{diam}(\Omega))^2}\,\lambda\ \operatorname{meas}(\{x\in\Omega_1\,:\,|\theta-\theta_g|>\lambda\})^{\frac{1}{3}} \leq c_{\mathrm{Korn}}\,\frac{L^2}{\eta^2}\,\|f(\theta)\|^2_{[L^2(\Omega_1)]^3}\,.$$

By Lemma A.2.2, we find that

$$\|\theta-\theta_g\|_{L^2(\Omega_1)} \leq \sqrt{2}\,c_{\mathrm{Korn}}\,c_0^2\,(1+\operatorname{diam}(\Omega))^2\,\frac{L^2\ \operatorname{meas}(\Omega_1)^{1/2}}{\eta^2\,\kappa_l}\,\|f(\theta)\|^2_{[L^2(\Omega_1)]^3}\,.$$

Now, in view of (6.1), we have, on the one hand $\|f(\theta)\|_{[L^2(\Omega_1)]^3} \leq |\vec{g}|\,\rho_1\,M_\theta\ \operatorname{meas}(\Omega_1)^{1/2}$, and on the other hand $\|f(\theta)\|_{[L^2(\Omega_1)]^3} \leq |\vec{g}|\,\rho_1\,\alpha\,\|\theta-\theta_M\|_{L^2(\Omega_1)}$. We obtain

$$\|\theta-\theta_M\|_{L^2(\Omega_1)} \leq \sqrt{2}\,c_{\mathrm{Korn}}\,c_0^2\,(1+\operatorname{diam}(\Omega))^2\,\frac{L^2\ \operatorname{meas}(\Omega_1)\,M_\theta\,|\vec{g}|^2\,\rho_1^2\,\alpha}{\eta^2\,\kappa_1}\,\|\theta-\theta_M\|_{L^2(\Omega_1)}\,.$$

The claim follows. □

Lemma 6.3.4. For $n\in\mathbb{N}$, consider an arbitrary sequence $\{j_{0,n},\,v_{g,n},\,\theta_{g,n}\}\subset X_{\theta_{\max}}$, and assume that $\hat{K}_{g,n} := \hat{K}_0(\{j_{0,n},\,v_{g,n},\,\theta_{g,n}\}) \to 0$ for $n\to\infty$.

Let $\theta_g \leq \theta_{\max}$ be the positive constant such that $\theta_{g,n}\to\theta_g$ in $W^{1,2}(\Omega)$, and assume in addition that the hypothesis of Theorem 6.3.1 are satisfied.

Then, for every $\{v_n,\,H_n,\,\theta_n\}\in S\big(\{j_{0,n},\,v_{g,n},\,\theta_{g,n}\}\big)$, and for every $1\leq r<3/2$, one has

$$\begin{aligned} v_n &\longrightarrow 0 \text{ in } D^{1,2}(\Omega_1)\,, \quad H_n \longrightarrow 0 \text{ in } \mathcal{H}_\mu(\tilde{\Omega})\,,\\ \theta_n &\longrightarrow \theta_g \text{ in } W^{1,r}(\Omega)\,, \quad \theta_n^4 \longrightarrow \theta_g^4 \text{ in } L^1(\Sigma)\,.\end{aligned}$$

Proof. Note first that every test function used in the proof of Proposition 6.1.3 for estimating θ has the form $g(\theta)$ with a continuous, bounded and increasing function g such that $g(0) = 0$. Thanks to Lemma 6.3.6, we can estimate θ_n in exactly the same way. Applying the techniques of Proposition 6.1.4, we then obtain the existence of $\{v,\, H,\, \theta\} \in S\big(\{0,\, 0,\, \theta_g\}\big)$ such that

$$v_n \longrightarrow v \text{ in } D^{1,2}(\Omega_1)\,, \quad H_n \longrightarrow H \text{ in } \mathcal{H}_\mu(\tilde{\Omega})\,, \quad \theta_n^4 \longrightarrow \theta^4 \text{ in } L^1(\Sigma)\,, \tag{6.56}$$

and for $1 \le r < 3/2$,

$$\theta_n \rightharpoonup \theta \text{ in } W^{1,r}(\Omega)\,. \tag{6.57}$$

Because of Lemma 6.3.3, we can verify that $\{v,\, H,\, \theta\} = \{0,\, 0,\, \theta_g\}$. Therefore, in view of (6.56), only the strong convergence

$$\theta_n \longrightarrow \theta_g \text{ in } W^{1,r}(\Omega)\,, \tag{6.58}$$

remains to prove. For some $\gamma \in]0,1[$, we test the relation (3.64) with the function

$$\xi_n := \operatorname{sign}(\theta_n - \theta_{g,n}) \left(1 - \frac{1}{(1 + |\theta_n - \theta_{g,n}|)^\gamma}\right).$$

By the result (6.56), (6.57), we easily prove that

$$\int_{\Omega_1} \rho_1\, c_V\, v_n \cdot \nabla\theta_n\, \xi_n \longrightarrow 0, \qquad \int_\Sigma G(\sigma\, \theta_n^4)\, \xi_n \longrightarrow 0\,,$$

$$\int_\Omega \Big(r\, |\operatorname{curl} H_n|^2 + \eta\, D(v_n, v_n)\Big)\, \xi_n \longrightarrow 0\,,$$

as $n \to \infty$. It follows that

$$\lim_{n\to\infty} \int_\Omega \kappa\, \gamma\, \frac{|\nabla\theta_n|^2}{(1 + |\theta_n - \theta_{g,n}|)^{1+\gamma}} = 0\,.$$

Therefore, we can find a subsequence such that

$$\frac{|\nabla\theta_n|^2}{(1 + |\theta_n - \theta_{g,n}|)^{1+\gamma}} \longrightarrow 0 \qquad \text{almost everywhere in } \Omega\,.$$

Since $(1 + |\theta_n - \theta_{g,n}|)^{1+\gamma} \longrightarrow 1$ almost everywhere in Ω, it follows that $|\nabla\theta_n| \longrightarrow 0$ almost everywhere in Ω. In view of (6.57) the sequence $\{\nabla\theta_n\}$ is bounded in the space $[L^r(\Omega)]^3$ for $1 \le r < 3/2$. Thus, we obtain for all $1 \le q < r < 3/2$ the strong convergence $\|\nabla\theta_n\|_{[L^q(\Omega)]^3} \longrightarrow 0$. Finally, we observe that the considerations of the present proof can be applied to each subsequence of the sequence $\{v_n,\, H_n,\, \theta_n\}$. This proves the claim. $\square$

Corollary 6.3.5. Under the assumptions of Theorem 6.3.1, it holds that

$$\lim_{\gamma\to 0} \sup_{\substack{\{v_g, j_0, \theta_g\} \in X_{\theta_{\max}} \\ \hat{K}_0(\{v_g, j_0, \theta_g\}) \le \gamma}} \; \sup_{\{v, H, \theta\} \in S\left(\{v_g, j_0, \theta_g\}\right)} \|\{v,\, H,\, \theta - \theta_g\}\|_Y = 0\,.$$

Proof. This is only a reformulation of the statement of Lemma 6.3.4. □

We now want to prove the main result of this section. Because of the numerous estimates involved, we split the proof into five steps.

Proof of Theorem 6.3.1. For some data $\{v_g,\, j_0,\, \theta_g\} \in X_{\theta_{\max}}$, assume that $\{v_1,\, H_1,\, \theta_1\}$ and $\{v_2,\, H_2,\, \theta_2\}$ both belong to the set $S\big(\{v_g,\, j_0,\, \theta_g\}\big)$ of solutions.

We define a number $S_0 > 0$ by

$$S_0 := \|\{v_1,\, H_1,\, \theta_1 - \theta_g\}\|_Y + \|\{v_2,\, H_2,\, \theta_2 - \theta_g\}\|_Y\,. \tag{6.59}$$

In view of Corollary 6.3.5, $\lim_{\hat{K}_0 \to 0} S_0 = 0$.

First step: estimates on $\mathbf{v_1 - v_2}$ and $\mathbf{H_1 - H_2}$.

Using the test functions $v_1 - v_2$ and $H_1 - H_2$ in the integral identities (3.62) and (3.63) written respectively for v_1, H_1 and v_2, H_2, we find

$$\begin{aligned}
\int_{\Omega_1} \eta\, D(v_1 - v_2\,,\, v_1 - v_2) &= -\int_{\Omega_1} \rho_1 \Big((v_1 \cdot \nabla) v_1 - (v_2 \cdot \nabla) v_2 \Big) \cdot (v_1 - v_2) \\
&\quad + \int_{\Omega_1} \Big(\big(\operatorname{curl} H_1 \times \mu\, H_1 \big) - \big(\operatorname{curl} H_2 \times \mu\, H_2 \big) \Big) \cdot (v_1 - v_2) \\
&\quad + \int_{\Omega_1} [f(\theta_1) - f(\theta_2)] \cdot (v_1 - v_2)\,,
\end{aligned}$$

and

$$\int_{\tilde{\Omega}} r\, |\operatorname{curl}(H_1 - H_2)|^2 = \int_{\Omega_1} \Big(\big(v_1 \times \mu\, H_1 \big) - \big(v_2 \times \mu\, H_2 \big) \Big) \cdot \operatorname{curl}(H_1 - H_2)\,.$$

We add both relations, and by straightforward rearrangements of terms, we get the estimate

$$\begin{aligned}
&\int_{\Omega_1} \eta\, D(v_1 - v_2\,,\, v_1 - v_2) + \int_{\tilde{\Omega}} r\, |\operatorname{curl}(H_1 - H_2)|^2 \\
&\quad \le \rho_1 \int_{\Omega_1} |\nabla v_2|\, |v_1 - v_2|^2 + \int_{\Omega_1} |f(\theta_1) - f(\theta_2)|\, |v_1 - v_2| \\
&\qquad + 2\,\mu_u \int_{\Omega_1} |H_1 - H_2| \Big(|\operatorname{curl} H_2|\, |v_1 - v_2| + |v_2|\, |\operatorname{curl}(H_1 - H_2)| \Big)\,.
\end{aligned}$$

We denote by c_0 the constant of the continuous embedding $W^{1,2}(\Omega_1) \hookrightarrow L^4(\Omega_1)$. Applying Lemma A.4.1, we obtain the estimate

$$\begin{aligned}
&\Big(\frac{\eta}{2} - \rho_1\, c_0^2\, \|\nabla v_2\|_{[L^2(\Omega_1)]^9} \Big) \int_{\Omega_1} D(v_1 - v_2\,,\, v_1 - v_2) \\
&\quad + \bigg(r_l - c_0^2\, c_{\mathcal{H}} \bigg[\rho_1\, \mu_u\, \|\nabla v_2\|_{[L^2(\Omega_1)]^9} - \frac{\mu_u^2}{\eta}\, \|\operatorname{curl} H_2\|^2_{[L^2(\tilde{\Omega})]^3} \bigg] \bigg) \int_{\tilde{\Omega}} |\operatorname{curl}(H_1 - H_2)|^2 \\
&\quad \le \frac{L^2}{\eta}\, \|f(\theta_1) - f(\theta_2)\|^2_{[L^2(\Omega_1)]^3}\,.
\end{aligned}$$

Using Corollary 6.3.5, we see that we can choose the number $\hat{K}_0$ as small as to achieve that

$$\Big(\frac{1}{2} - \frac{\rho_1\, c_0^2\, S_0}{\eta}\Big) > 0\,, \qquad \Big(1 - c_0^2\, c_{\mathcal{H}}\, \frac{1}{r_l}\, \Big[\rho_1\, \mu_u\, S_0 - \frac{\mu_u^2}{\eta}\, S_0^2\Big]\Big) > 0\,.$$

Setting

$$\beta := \max\Big\{\Big(\frac{1}{2} - \frac{\rho_1\, c_0^2\, S_0}{\eta}\Big)^{-1}\,,\ \Big(1 - c_0^2\, c_{\mathcal{H}}\, \frac{1}{r_l}\, \Big[\rho_1\, \mu_u\, S_0 - \frac{\mu_u^2}{\eta}\, S_0^2\Big]\Big)^{-1}\Big\}\,,$$

we can write

$$\int_{\Omega_1} \eta\, D(v_1 - v_2\,,\, v_1 - v_2) + \int_{\tilde{\Omega}} r\,|\,\mathrm{curl}(H_1 - H_2)|^2 \leq \frac{L^2\,\beta}{\eta}\, \|f(\theta_1) - f(\theta_2)\|^2_{[L^2(\Omega_1)]^3}\,. \tag{6.60}$$

Second step: estimates of volume integrals involving $\theta_1 - \theta_2$.
For $\lambda > 0$ arbitrary, we now consider the test function

$$\xi_\lambda := (\theta_1 - \theta_2)^{(\lambda)} := \mathrm{sign}(\theta_1 - \theta_2)\, \min\{|\theta_1 - \theta_2|\,,\, \lambda\}\,.$$

We substract the integral identities (3.64), written respectively for θ_1 and θ_2. In view of Lemma 6.3.6, we can test the resulting relation with ξ_λ, and obtain

$$\begin{aligned}
&\int_\Omega \rho_1\, c_V\, \Big(v_1 \cdot \nabla\theta_1 - v_2 \cdot \nabla\theta_2\Big)\, (\theta_1 - \theta_2)^{(\lambda)} + \int_\Omega \kappa\, |\nabla(\theta_1 - \theta_2)^{(\lambda)}|^2 \\
&\qquad + \int_\Sigma G\big(\sigma\, [\theta_1^4 - \theta_2^4]\big)\, (\theta_1 - \theta_2)^{(\lambda)} \\
&\quad \leq \int_\Omega \Big(r\, \big(|\,\mathrm{curl}\, H_1|^2 - |\,\mathrm{curl}\, H_2|^2\big) + \eta\, \big(D(v_1\,,\, v_1) - D(v_2\,,\, v_2)\big)\Big)\, (\theta_1 - \theta_2)^{(\lambda)}\,.
\end{aligned} \tag{6.61}$$

We want to estimate each term appearing in this relation. Since by the usual arguments $\int_{\Omega_1} \rho_1\, c_V\, v_1 \cdot \nabla(\theta_1 - \theta_2)\, (\theta_1 - \theta_2)^{(\lambda)} = 0$, we have

$$\begin{aligned}
\Big|\int_\Omega \rho_1\, c_V\, \Big(v_1 \cdot \nabla\theta_1 - v_2 \cdot \nabla\theta_2\Big)\, (\theta_1 - \theta_2)^{(\lambda)}\Big| &= \Big|\int_\Omega \rho_1\, c_V\, (v_1 - v_2) \cdot \nabla\theta_2\Big)\, (\theta_1 - \theta_2)^{(\lambda)}\Big| \\
&\leq \rho_1\, c_V\, \|v_1 - v_2\|_{[L^4(\Omega_1)]^3}\, \|\nabla\theta_2\|_{L^{4/3}(\Omega_1)}\, \lambda\,.
\end{aligned} \tag{6.62}$$

By the triangle inequality, we can write

$$\|\nabla\theta_2\|_{L^{4/3}(\Omega_1)} \leq \|\nabla(\theta_2 - \theta_0)\|_{L^{4/3}(\Omega_1)} + \|\nabla\theta_0\|_{L^{4/3}(\Omega_1)} \leq S_0 + \hat{K}_0\,.$$

In view of (6.62) and (6.60), we then can write

$$\begin{aligned}
&\Big|\int_\Omega \rho_1\, c_V\, \Big(v_1 \cdot \nabla\theta_1 - v_2 \cdot \nabla\theta_2\Big)\, (\theta_1 - \theta_2)^{(\lambda)}\Big| \\
&\qquad \leq c_0\, (1 + \mathrm{diam}(\Omega_1))\, c_{\mathrm{Korn}}\, \frac{\rho_1\, c_V\, L\, \sqrt{\beta}}{\eta}\, (S_0 + \hat{K}_0)\, \|f(\theta_1) - f(\theta_2)\|_{[L^2(\Omega_1)]^3}\, \lambda\,.
\end{aligned} \tag{6.63}$$

Turning our attention to the term on the right-hand side of (6.61), we have in view of (6.60) that

$$\begin{aligned}
&\left| \int_\Omega \Big(r \left(|\operatorname{curl} H_1|^2 - |\operatorname{curl} H_2|^2 \right) + \eta \left(D(v_1\,,\, v_1) - D(v_2\,,\, v_2) \right) \Big) \, (\theta_1 - \theta_2)^{(\lambda)} \right| \\
&\quad \le \left\| r^{1/2} \operatorname{curl}(H_1 + H_2) \right\|_{[L^2(\tilde\Omega)]^3} \left\| r^{1/2} \operatorname{curl}(H_1 - H_2) \right\|_{[L^2(\tilde\Omega)]^3} \lambda \\
&\qquad + \left\| \eta^{1/2} \, D(v_1 + v_2, v_1 + v_2) \right\|_{[L^2(\Omega_1)]^9} \left\| \eta^{1/2} \, D(v_1 - v_2, v_1 - v_2) \right\|_{[L^2(\Omega_1)]^9} \lambda \\
&\quad \le \sqrt{r_u + \eta_u} \, S_0 \, \frac{L\sqrt{\beta}}{\sqrt{\eta}} \, \| f(\theta_1) - f(\theta_2) \|_{[L^2(\Omega_1)]^3} \, \lambda \,. \qquad (6.64)
\end{aligned}$$

Third step: estimates of surface integrals involving $\boldsymbol{\theta_1 - \theta_2}$.
Using the fact that G is selfadjoint, we first write

$$\begin{aligned}
\int_\Sigma G\big(\sigma \, [\theta_1^4 - \theta_2^4]\big) \, (\theta_1 - \theta_2)^{(\lambda)} &= \int_\Sigma \sigma \, [\theta_1^4 - \theta_2^4] \, G\big((\theta_1 - \theta_2)^{(\lambda)}\big) \\
&= \int_\Sigma \sigma \, (\theta_1^2 + \theta_2^2) \, (\theta_1 + \theta_2) \, (\theta_1 - \theta_2) \, G\big((\theta_1 - \theta_2)^{(\lambda)}\big) \,.
\end{aligned}$$

With the abbreviations $F(\theta_1, \theta_2) := (\theta_1^2 + \theta_2^2)\,(\theta_1 + \theta_2)$ and $F_0 = F(\theta_g,\, \theta_g)$, we can also write

$$\int_\Sigma G\big(\sigma \, [\theta_1^4 - \theta_2^4]\big) \, (\theta_1 - \theta_2)^{(\lambda)} = \int_\Sigma \sigma \, F(\theta_1, \theta_2) \, (\theta_1 - \theta_2) \, G\big((\theta_1 - \theta_2)^{(\lambda)}\big) \,.$$

Using the decomposition $G = I - \mathbf{H}$, it is easy to prove the inequality

$$\begin{aligned}
\int_\Sigma \sigma \, F(\theta_1, \theta_2) \, (\theta_1 - \theta_2) \, G\big((\theta_1 - \theta_2)^{(\lambda)}\big) &\ge \int_\Sigma \sigma \, F(\theta_1, \theta_2) \, (\theta_1 - \theta_2)^{(\lambda)} \, G\big((\theta_1 - \theta_2)^{(\lambda)}\big) \\
= \int_\Sigma \sigma \, \big[F(\theta_1, \theta_2) - F_0 \big] \, (\theta_1 - \theta_2)^{(\lambda)} \, G\big((\theta_1 - \theta_2)^{(\lambda)}\big) & \\
+ \int_\Sigma \sigma \, F_0 \, (\theta_1 - \theta_2)^{(\lambda)} \, G\big((\theta_1 - \theta_2)^{(\lambda)}\big) \,. & \qquad (6.65)
\end{aligned}$$

We want to estimate from below the right-hand side of (6.65). By Lemma 2, we have

$$\begin{aligned}
&\int_\Sigma \sigma \, \big[F(\theta_1, \theta_2) - F_0 \big] \, (\theta_1 - \theta_2)^{(\lambda)} \, G\big((\theta_1 - \theta_2)^{(\lambda)}\big) \\
&\quad = \int_\Sigma \epsilon \, \sigma \, \big[F(\theta_1, \theta_2) - F_0 \big] \, |(\theta_1 - \theta_2)^{(\lambda)}|^2 \\
&\qquad - \int_\Sigma \epsilon \, \sigma \, \big[F(\theta_1, \theta_2) - F_0 \big] \, (\theta_1 - \theta_2)^{(\lambda)} \, \tilde{\mathbf{H}}\big((\theta_1 - \theta_2)^{(\lambda)}\big) \,. \qquad (6.66)
\end{aligned}$$

We introduce a measurable set $A_0 \subseteq \Sigma$ defined by

$$A_0 := \{ z \in \Sigma \,:\, F(\theta_1, \theta_2) - F_0 \ge 0 \} \,.$$

We can decompose

$$\int_{\Sigma} \epsilon\,\sigma\,\big[F(\theta_1,\theta_2) - F_0\big]\,|(\theta_1-\theta_2)^{(\lambda)}|^2$$
$$= \int_{A_0} \epsilon\,\sigma\,\big[F(\theta_1,\theta_2) - F_0\big]\,|(\theta_1-\theta_2)^{(\lambda)}|^2 + \int_{A_0^c} \epsilon\,\sigma\,\big[F(\theta_1,\theta_2) - F_0\big]\,|(\theta_1-\theta_2)^{(\lambda)}|^2 \,.$$

Observe that the term to the left is positive, we do not need to consider it anymore in the following estimates. For all $1 < p < 4$, we choose p' such that $1/p + 1/p' = 1$ and we can estimate

$$\left|\int_{A_0^c} \epsilon\,\sigma\,\big[F(\theta_1,\theta_2) - F_0\big]\,|(\theta_1-\theta_2)^{(\lambda)}|^2\right| \le \sigma\,\lambda\,\|\theta_1-\theta_2\|_{L^p(\Sigma)}\,\left(\int_{A_0^c} |F(\theta_1,\theta_2)-F_0|^{p'}\right)^{\frac{1}{p'}}$$
$$\le \sigma\,\lambda\,\|\theta_1-\theta_2\|_{L^p(\Sigma)}\,(2\,F_0)^{\frac{p'-4/3}{p'}}\,\left(\int_{A_0^c} |F(\theta_1,\theta_2)-F_0|^{4/3}\right)^{\frac{1}{p'}}$$
$$\le \sigma\,\lambda\,\|\theta_1-\theta_2\|_{L^p(\Sigma)}\,[2\,F(\theta_{\max},\,\theta_{\max})]^{\frac{p'-4/3}{p'}}\,\|F(\theta_1,\theta_2)-F_0\|_{L^{4/3}(\Sigma)}^{\frac{4}{3\,p'}} \,. \tag{6.67}$$

For the second term in (6.66), we have the estimate

$$\left|\int_{\Sigma} \epsilon\,\sigma\,\big[F(\theta_1,\theta_2) - F_0\big]\,(\theta_1-\theta_2)^{(\lambda)}\,\tilde{\mathbf{H}}\big((\theta_1-\theta_2)^{(\lambda)}\big)\right|$$
$$\le \sigma\,\lambda\,\|\epsilon^{3/4}\,(F(\theta_1,\theta_2)-F_0)\|_{L^{4/3}(\Sigma)}\,\|\epsilon^{1/4}\,\tilde{\mathbf{H}}\big((\theta_1-\theta_2)^{(\lambda)}\big)\|_{L^4(\Sigma)} \,. \tag{6.68}$$

Observe that according to Lemma 4.1.9, the operator $\tilde{H}$ belongs to $\mathcal{L}(L^{\tilde{p}}(\Sigma), C(\Sigma))$ for a $\tilde{p} < 2$. In view of the assumption (6.51), we therefore have

$$\|\epsilon^{1/4}\,\tilde{\mathbf{H}}\big((\theta_1-\theta_2)^{(\lambda)}\big)\|_{L^4(\Sigma)} \le \text{meas}(\Sigma)^{\frac{1}{4}}\,\|\tilde{\mathbf{H}}\big((\theta_1-\theta_2)^{(\lambda)}\big)\|_{C(\Sigma)}$$
$$\le c\,\text{meas}(\Sigma)^{\frac{1}{4}}\,\|\theta_1-\theta_2\|_{L^{\tilde{p}}(\Sigma)} \,. \tag{6.69}$$

Summarizing the results (6.67), (6.68), (6.69),

$$\int_{\Sigma} \sigma\,\big[F(\theta_1,\theta_2) - F_0\big]\,(\theta_1-\theta_2)^{(\lambda)}\,G\big((\theta_1-\theta_2)^{(\lambda)}\big) \ge -f(\hat{K}_0)\,\|\theta_1-\theta_2\|_{L^{\tilde{p}}(\Sigma)}\,\lambda \,,$$

with

$$f(\hat{K}_0) \le \tilde{c}\,(\|F(\theta_1,\theta_2)-F_0\|_{L^{4/3}(\Sigma)}^{\frac{4}{3\,\tilde{p}'}} + \|(F(\theta_1,\theta_2)-F_0)\|_{L^{4/3}(\Sigma)}) \to 0 \,,$$

as $\hat{K}_0$ converges to zero.

The second term on the right-hand side of (6.65) can be estimated in quite similar matter, so that we finally obtain

$$\int_{\Sigma} G\big(\sigma\,[\theta_1^4 - \theta_2^4]\big)\,(\theta_1-\theta_2)^{(\lambda)} \ge -\tilde{f}(\hat{K}_0)\,\lambda\,\|\theta_1-\theta_2\|_{L^{\tilde{p}}(\Sigma)} \,, \tag{6.70}$$

with $\tilde{p} < 2$ and a sequence of numbers $\tilde{f}(\hat{K}_0)$ which converges to zero together with $\hat{K}_0$.

Fourth Step: final estimate

The results (6.63), (6.64) and (6.70) give the inequality

$$\int_\Omega \kappa\, |\nabla(\theta_1-\theta_2)^{(\lambda)}|^2 \le c_1\, \bar{f}(\hat{K}_0)\,(\|f(\theta_1)-f(\theta_2)\|_{[L^2(\Omega_1)]^3} + \|\theta_1-\theta_2\|_{L^{\tilde{p}}(\Sigma)})\,\lambda\,. \tag{6.71}$$

By the inequality (6.71), we obtain that

$$\big\|(\theta_1-\theta_2)^{(\lambda)}\big\|^2_{W^{1,2}_\Gamma(\Omega)} \le \bar{C}\ \bar{f}(\hat{K}_0)\,(\|f(\theta_1)-f(\theta_2)\|_{[L^2(\Omega_1)]^3} + \|\theta_1-\theta_2\|_{L^{\tilde{p}}(\Sigma)})\,\lambda\,. \tag{6.72}$$

Using the continuity of the embedding $W^{1,2}_\Gamma(\Omega) \hookrightarrow L^6(\Omega)$ and $W^{1,2}_\Gamma(\Omega) \hookrightarrow L^4(\Sigma)$, it follows that

$$\begin{aligned}
\lambda^2 \operatorname{meas}\Big(\{x\in\Omega\,:\,|\theta_1-\theta_2|>\lambda\}\Big)^{1/3} &+ \lambda^2 \operatorname{meas}\Big(\{z\in\Sigma\,:\,|\theta_1-\theta_2|>\lambda\}\Big)^{1/2}\\
&\le \big\|(\theta_1-\theta_2)^{(\lambda)}\big\|^2_{L^6(\Omega)} + \big\|(\theta_1-\theta_2)^{(\lambda)}\big\|^2_{L^4(\Sigma)}\\
&\le c\,\bar{C}\,\bar{f}(\hat{K}_0)\,(\|f(\theta_1)-f(\theta_2)\|_{[L^2(\Omega_1)]^3} + \|\theta_1-\theta_2\|_{L^{\tilde{p}}(\Sigma)})\,\lambda\,.
\end{aligned}$$

On the other hand, we can use the the result of Lemma A.2.2 to obtain that

$$\begin{aligned}
\big\|\theta_1-\theta_2\big\|_{L^2(\Omega)} + \|\theta_1-\theta_2\|_{L^{\tilde{p}}(\Sigma)} \le c\,&\left(\sup_{\lambda>0}\left\{\lambda \operatorname{meas}\Big(\{x\in\Omega\,:\,|\theta_1-\theta_2|>\lambda\}\Big)^{1/3}\right\}\right.\\
&\left.+\sup_{\lambda>0}\left\{\lambda \operatorname{meas}\Big(\{z\in\Sigma\,:\,|\theta_1-\theta_2|>\lambda\}\Big)^{1/2}\right\}\right)\,.
\end{aligned}$$

The estimate (6.72) finally yields

$$\begin{aligned}
\big\|\theta_1-\theta_2\big\|_{L^2(\Omega)} + \big\|\theta_1-\theta_2\big\|_{L^{\tilde{p}}(\Sigma)} &\le C\,\bar{f}(\hat{K}_0)\,(\|f(\theta_1)-f(\theta_2)\|_{[L^2(\Omega_1)]^3} + \|\theta_1-\theta_2\|_{L^{\tilde{p}}(\Sigma)})\\
&\le\ C\,\bar{f}(\hat{K}_0)\,(\rho_1\,|\vec{g}|\,\alpha\,\|\theta_1-\theta_2\|_{L^2(\Omega_1)} + \|\theta_1-\theta_2\|_{L^{\tilde{p}}(\Sigma)})
\end{aligned} \tag{6.73}$$

The claim follows. □

Lemma 6.3.6. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with the structure described in the paragraph 3.1. Assume that for $i = 0,\dots,m$, the boundary $\partial\Omega_i \setminus \partial\Omega$ belongs to $\mathcal{C}^1$ and that the outher boundary $\partial\Omega$ belongs to $\mathcal{C}^{0,1}$. For $i = 0,\dots,m$, let κ_i be constant. Let $f \in L^1(\Omega)$ and $h \in L^1(\Sigma)$ be given.

Then, there exists a unique $u \in W^{1,p}_0(\Omega)$, $p < 3/2$ arbitrary, such that

$$\int_\Omega \kappa\,\nabla u\cdot\nabla\xi = \int_\Omega f\,\xi + \int_\Sigma h\,\xi\,, \tag{6.74}$$

for all $\xi \in W^{1,p'}_0(\Omega)$.

In addition, for every monotonely increasing, bounded, real-valued function g with $g(0) = 0$, one has

$$\int_\Omega \kappa\, g'(u)\,|\nabla u|^2 \le \int_\Omega f\,g(u) + \int_\Sigma h\,g(u)\,, \tag{6.75}$$

Proof. Existence in the class $\bigcap_{1\leq p<3/2} W_0^{1,p}(\Omega)$ was already proved in Stampacchia [1965], and confirmed for example in Rakotoson [1991]. The solution is also unique. Suppose that u_1, u_2 both satisfy (6.74). Then the difference satisfies

$$\int_\Omega \kappa\, \nabla(u_1 - u_2)\cdot\nabla\xi = 0\,,$$

for all $\xi \in W^{1,r}(\Omega)$ with $r > 3$. Under the assumptions of the lemma, the main theorem of the paper Elschner et al. [2007] gives the existence of a $q > 3$ such that $u_1 - u_2 \in W_0^{1,q}(\Omega)$. The uniqueness clearly follows.

We prove the last claim. For $\delta > 0$, consider the function $u_\delta \in W_0^{1,2}(\Omega)$ that satisfies

$$\int_\Omega \kappa\, \nabla u_\delta\cdot\nabla\xi = \int_\Omega [f]_{(\delta)}\,\xi + \int_\Sigma [h]_\delta\,\xi\,, \tag{6.76}$$

for all $\xi \in W_0^{1,2}(\Omega)$. By the usual uniform estimates for linear elliptic problems with L^1−right-hand sides, we find for all $1 \leq p < 3/2$ a subsequence such that

$$u_\delta \rightharpoonup w \ \text{ in } W_0^{1,p}(\Omega)\,, \quad u_\delta \longrightarrow w \ \text{ in } L^p(\Omega)\,, \tag{6.77}$$

as $\delta \to 0$. By the uniqueness of u obtained above, $w = u$. Let $g : \mathbb{R} \to \mathbb{R}$ be monotonely increasing and bounded such that $g(0) = 0$. We test the relation (6.76) with $\xi = g(u_\delta)$ and obtain

$$\int_\Omega \kappa\, g'(u_\delta)\,|\nabla u_\delta|^2 = \int_\Omega [f]_{(\delta)}\, g(u_\delta) + \int_\Sigma [h]_\delta\, g(u_\delta)\,.$$

Since g' is positive, we can introduce the function

$$F(s) := \int_0^s \sqrt{g'(\tau)}\,d\tau\,,$$

and write

$$\int_\Omega \kappa\, |\nabla F(u_\delta)|^2 = \int_\Omega [f]_{(\delta)}\, g(u_\delta) + \int_\Sigma [h]_\delta\, g(u_\delta)\,. \tag{6.78}$$

Clearly, by the last relation,

$$F(u_\delta) \rightharpoonup \tilde{w} \quad \text{in } W_0^{1,2}(\Omega)\,,$$

for a subsequence. But in view of (6.77), we immediately find that $\tilde{w} = F(u)$.

Passing to the limit in (6.78) for this subsequence, we obtain the inequality (6.75), and the lemma is proved. □

6.4 A regularity result

We recall the following result proved in the section 5. Let $1 < \alpha,\, p < \infty$. For a bounded Lipschitz domain $U \subset \mathbb{R}^3$, we consider the space (cp. (5.8))

$$\mathcal{W}^{p,\alpha}(U) := \left\{ u \in L^p_{\text{curl}}(U) \cap L^p_{\text{div}}(U) \,\middle|\, u\cdot\vec{n} \in L^\alpha(\partial U) \right\}.$$

Define the Sobolev embedding exponent p^* by

$$p^* := \begin{cases} \frac{3p}{3-p} & \text{if } p < 3\,, \\ 1 \le s < \infty \text{ arbitrary} & \text{if } p = 3\,, \\ \infty & \text{if } p > 3\,. \end{cases}$$

We recall the result of Proposition 5.1.2.

Theorem 6.4.1. Let the assumption of Theorem 6.1.1 be satisfied, but assume in addition that the function $\mathfrak{s}_1$ of electrical conductivity in the fluid is a function of the position that belongs to $C^1(\overline{\tilde{\Omega}_1})$, and that the reinforced assumption (3.50) is satisfied in connection with (5.17).

Then, there exists a weak solution of (P_{st}) such that

(1) The vector field $\operatorname{curl} H$ belongs to the space $\mathcal{W}^{\frac{3}{2},\infty}(\tilde{\Omega}_1)$. In particular, $\operatorname{curl} H \in [L^3(\Omega_1)]^3$, and there exists a constant $C = C(\tilde{\Omega},\, \mathfrak{s}_1)$ such that

$$\|\operatorname{curl} H\|_{[L^3(\Omega_1)]^3} \le C\left(\|v\|_{[W^{1,2}(\Omega_1)]^3}\,\|H\|_{[W^{1,2}(\Omega_1)]^3} + \|\operatorname{curl} H\|_{[L^{3/2}(\Omega_1)]^3}\right).$$

(2) If in addition the function η is a smooth function of the position in Ω_1, the temperature θ belongs to the space $V^{2,5}(\Omega) \cap L^\infty(\Omega)$.

Proof. (1): Observe first that under the assumptions of Theorem 6.1.1, we have by Lemma 5.2.3, (1) for $i = 0, \ldots, m$ that

$$H \in [W^{1,2}(\tilde{\Omega}_i)]^3\,, \tag{6.79}$$

whenever $\{v,\, H,\, \theta\}$ is a weak solution of (P_{st}). Using the formula (A.11) and the fact that v and $\mu\, H$ are divergence free, we can therefore write almost everywhere in Ω_1 that

$$\operatorname{curl}(v \times \mu\, H) = \big(\mu\, H \cdot \nabla\big)v - \big(v \cdot \nabla\big)(\mu\, H) = \big(\mu\, H \cdot \nabla\big)v - \mu\,\big(v \cdot \nabla\big)H - \,v \cdot \nabla\mu\, H\,.$$

By means of Sobolev's embedding theorems we get

$$\begin{aligned} \|\operatorname{curl}(v \times \mu\, H)\|_{[L^{3/2}(\Omega_1)]^3} &\le \mu_u\,\|\nabla v\|_{[L^2(\Omega_1)]^9}\,\|H\|_{[L^6(\Omega_1)]^3} + \mu_u\,\|\nabla H\|_{[L^2(\Omega_1)]^9}\,\|v\|_{[L^6(\Omega_1)]^3} \\ &\quad + \|\mu\|_{C^1(\overline{\Omega_1})}\,\|H\|_{[L^3(\Omega_1)]^3}\,\|v\|_{[L^3(\Omega_1)]^3} \\ &\le c\,\|v\|_{[W^{1,2}(\Omega_1)]^3}\,\|H\|_{[W^{1,2}(\Omega_1)]^3}\,. \end{aligned} \tag{6.80}$$

If $\{v,\, H,\, \theta\}$ is a weak solution of (P_{st}), then the relation

$$\int_{\tilde{\Omega}} r\, \operatorname{curl} H \cdot \operatorname{curl}\psi = \int_{\Omega_1} (v \times \mu\, H) \cdot \operatorname{curl}\psi\,, \tag{6.81}$$

is valid for all $\psi \in \mathcal{H}^0_\mu(\tilde{\Omega})$. With arguments similar to Lemma A.4.3, we readily can show that (6.81) even holds for all $\psi \in [C_c^\infty(\Omega_c)]^3$.

We in particular choose $\psi \in [C_c^\infty(\Omega_1)]^3$ arbitrary. We can integrate by parts the right-hand side of (6.81) to obtain that

$$\int_{\Omega_1} r \operatorname{curl} H \cdot \operatorname{curl} \psi = \int_{\Omega_1} \operatorname{curl}(v \times \mu\, H) \cdot \psi\,.$$

This means that $\operatorname{curl}(r \operatorname{curl} H) = \operatorname{curl}(v \times \mu\, H)$ in the sense of the generalized curl operator in Ω_1.

Define $w := r \operatorname{curl} H$. In view of (6.80), we can write in the generalized sense of the operator curl that

$$\operatorname{curl} w = \operatorname{curl}(v \times \mu\, H) \in [L^{3/2}(\Omega_1)]^3\,. \tag{6.82}$$

On the other hand, since $r = 1/\mathfrak{s} \in C^1(\overline{\Omega_1})$, we easily compute that

$$\operatorname{div} w = \nabla r \cdot \operatorname{curl} H \in L^2(\Omega_1)\,. \tag{6.83}$$

Finally, we know from Lemma A.4.2, (1) that

$$w \cdot \vec{n} = 0 \text{ on } \partial\Omega_1\,. \tag{6.84}$$

In view of (6.82), (6.83) and (6.84), we obtain that $w \in \mathcal{W}^{\frac{3}{2},\infty}(\Omega_1)$. Applying Lemma 5.1.2 with $\alpha = \infty$ and $p = 3/2$, we prove that

$$\begin{aligned}\|w\|_{[L^3(\Omega_1)]^3} &\leq c\,(\|\operatorname{curl} w\|_{[L^{3/2}(\Omega_1)]^3} + \|\operatorname{div} w\|_{L^{3/2}(\Omega_1)} + \|w \cdot \vec{n}\|_{L^\infty(\partial\Omega_1)})\\ &\leq C\,(\|v\|_{[W^{1,2}(\Omega_1)]^3}\, \|H\|_{[W^{1,2}(\Omega_1)]^3} + \|\operatorname{curl} H\|_{[L^{3/2}(\Omega_1)]^3})\,.\end{aligned}$$

(2): If the viscosity η is smooth, then the classical regularity theory for the Navier-Stokes equations (see Temam [1977], Ch. 2, Paragraph 1) gives that $v \in [W^{2,2}(\Omega_1)]^3$. This allows to estimate in (6.80) the L^2-norm of $\operatorname{curl}(v \times \mu\, H)$. We obtain that $H \in \mathcal{W}^{2,\infty}(\Omega_1)$. By Sobolev's embedding theorems and Lemma 5.1.2, the right-hand side of the heat equation is given by

$$r\, |\operatorname{curl} H|^2 + \chi_{\Omega_1}\, \eta\, D(v, v) \in [L^3(\Omega)]^3\,.$$

We only have to apply the results of Druet [2009a] about the solution operator of the heat equation with nonlocal radiation terms in order to prove the claim. $\square$

Chapter 7

The initial boundary value problem for the time-dependent system

In this chapter, we study the problem described in the paragraph 3.1.2.

7.1 An existence result

We begin formulating some additional assumptions, valid throughout the present section, with respect to the force f given by (3.23). Note that $\theta_M = \theta_M(t)$, where $\theta_M(t)$ denotes the mean value of $\theta(t)$ taken over the set Ω_1 (this ensures conservation of mass). Boussinesq's approximation is valid only in the range of small density variations, $\alpha\,(\theta - \theta_M) \ll 1$. For the sake of simplicity, we therefore replace in this section (3.23) by

$$f(\theta) = -\rho_1\,\vec{g}\,\mathrm{sign}(\theta - \theta_M)\,\min\{\alpha\,|\theta - \theta_M|, M_\theta\}\,, \tag{7.1}$$

with a positive number M_θ which can be interpreted as the maximal density fluctuations allowed in the fluid. Observe that the function $f : \mathbb{R} \longrightarrow \mathbb{R}^3$ defined by (7.1) is continuous and satisfies

$$|f(s)| \leq \rho_1\,|\vec{g}|\,M_\theta \quad \text{for all } s \in \mathbb{R}\,. \tag{7.2}$$

The genuine Boussinesq model is treated in the next section.

We further assume that the boundary data $v_g \in D^1_2(Q_1)$ is such that

$$\|v_g\|_{[V^1_2(Q_1)]^3} + \|v_g\|_{[L^\infty(Q_1)]^3} < +\infty\,, \tag{7.3}$$

where

$$\|v_g\|_{[V^1_2(Q_1)]^3} := \operatorname*{ess\,sup}_{t\in]0,T[} \|v_g(t)\|_{[L^2(\Omega_1)]^3} + \left\|\frac{\partial v_g}{\partial t}\right\|_{[L^2(Q_1)]^3} + \|\nabla v_g\|_{[L^2(Q_1)]^9}\,.$$

At time zero, we assume that the condition $v_g(0) = v_0$ in Ω_1 is satisfied, which simplifies homogenization matters. In order to avoid technical complications with respect to temperature homogenization, we assume that there exists a positive constant c such that

$$\theta_0 = c \text{ in } \Omega\,, \quad \theta_g = c \text{ on } \mathcal{C}\,. \tag{7.4}$$

However, we can extend our results to the case of general initial and boundary data θ_0, θ_g that fulfill the compatibility condition $\theta_g(0) = \theta_0$ in Ω and the requirement $\theta_0 = \text{const}$ on Σ.

We can state a general existence result.

Theorem 7.1.1. Assume that the hypotheses of the section 3.1.3 are satisfied, but not necessarily (3.50) or (3.51). Let the force f be given by (7.1), and let the boundary and initial data satisfy (7.3) and (7.4).

Then there exists at least one weak solution to (P) with defect measure, in the sense of Definition 3.2.4.

The remainder of this section is devoted to the proof of Theorem 7.1.1. For some $p \geq 5$ fixed, we work with the auxiliary spaces

$$\mathcal{V} := L^2(0,T;D^{1,2}(\Omega_1)) \times L^2(0,T;\mathcal{H}_\mu(\tilde{\Omega})) \times L^p(0,T;W^{1,p}(\Omega))\,,$$
$$\mathcal{V}_0 := L^2(0,T;D_0^{1,2}(\Omega_1)) \times L^2(0,T;\mathcal{H}_\mu(\tilde{\Omega})) \times L^p(0,T;W_\Gamma^{1,p}(\Omega))\,.$$

We have the isometry

$$\mathcal{V}_0^* \cong L^2(0,T;[D_0^{1,2}(\Omega_1)]^*) \times L^2(0,T;[\mathcal{H}_\mu(\tilde{\Omega})]^*) \times L^{p'}(0,T;[W_\Gamma^{1,p}(\Omega)]^*)\,,$$

where p' is defined by the relation $1/p + 1/p' = 1$. Observe that

$$D_0^{1,2}(\Omega_1) \overset{i_1}{\hookrightarrow} [L^2(\Omega_1)]^3 \overset{i_2}{\hookrightarrow} [D_0^{1,2}(\Omega_1)]^*\,,$$

with continuous injections i_1, i_2. Therefore, every element $v \in L^2(0,T;D_0^{1,2}(\Omega_1))$ has a well-defined distributional time derivative $v' \in L^2(0,T;[D_0^{1,2}(\Omega_1)]^*)$. Similarly, there are continuous injections j_1, j_2 and k_1, k_2 such that

$$\mathcal{H}_\mu(\tilde{\Omega}) \overset{j_1}{\hookrightarrow} [L^2(\tilde{\Omega})]^3 \overset{j_2}{\hookrightarrow} [\mathcal{H}_\mu(\tilde{\Omega})]^*\,,$$
$$W_\Gamma^{1,p}(\Omega) \overset{k_1}{\hookrightarrow} L^p(\Omega) \overset{k_2}{\hookrightarrow} [W_\Gamma^{1,p}(\Omega)]^*\,.$$

We are thus allowed to regard the distributional time derivative of $H \in L^2(0,T;\mathcal{H}_\mu(\tilde{\Omega}))$ as an element $H' \in L^2(0,T;[\mathcal{H}_\mu(\tilde{\Omega})]^*)$, and analogously for $\theta \in L^p(0,T;W_\Gamma^{1,p}(\Omega))$, $\theta' \in L^{p'}(0,T;[W_\Gamma^{1,p}(\Omega)]^*)$. We set

$$D(L) := \Big\{\{\phi,\psi,\xi\} \in \mathcal{V}_0 \,\Big|\, \exists\,\{\phi',\,\psi',\,\xi'\} \in \mathcal{V}_0^*\,;\ \{\phi(0),\psi(0),\xi(0)\} = 0\Big\}\,.$$

By canonical results that can be found in Lions [1969], Ch. 3, Lemma 1.1, the space $D(L)$ is dense in $\mathcal{V}_0$, and the mapping $L\big(\{\phi,\,\psi,\,\xi\}\big) = \{\phi',\,\psi',\,\xi'\}$ is linear and maximal monotone from $D(L)$ into $\mathcal{V}_0^*$.

For $\delta > 0$, we use the following notations. For vector fields $v : Q_1 \longrightarrow \mathbb{R}^3$ with vanishing divergence, we will denote by $(v)_{(\delta)}$ a mollifier of v such that

$$\operatorname{div}(v)_{(\delta)} = 0 \text{ in } Q_1, \qquad (v)_{(\delta)} \longrightarrow v \text{ in } [W_2^{1,0}(Q_1)]^3 \quad \text{as } \delta \to 0\,. \tag{7.5}$$

For functions $g : \tilde{Q} \longrightarrow \mathbb{R}$ we denote by $[g]_{(\delta)}$ a cutoff operator of g, for example the operator (6.9). For real valued functions u defined in the space-time cylinder $]0,T[\times\tilde{\Omega}$, and for parameters $h \in]0,T[$, we recall the notation

$$u_{(h)}(x,t) := \frac{1}{h}\int_t^{t+h} u(x,\tau)\,d\tau\,.$$

The averaging operator $(\cdot)_{(h)}$ is called *Steklov averaging* of a function. Its essential properties are listed in Ladyzenskaja et al. [1968]), Ch. 2, Section 4.

Proposition 7.1.2. Let $\delta > 0$ and $p \geq 5$ be fixed real numbers. If the assumptions of Theorem 6.1.1 are satisfied, then there exists a triple $\{v_\delta,\, H_\delta,\, \theta_\delta\} \in \mathcal{V}$ such that

$$\begin{aligned} v_\delta = v_g \text{ on }]0,T[\times\partial\Omega_1 \text{ and in } \{0\}\times\Omega_1\,, \quad H_\delta = 0 \text{ in } \{0\}\times\tilde{\Omega}, \\ \theta_\delta = \theta_g \text{ on }]0,T[\times\partial\Omega \text{ and in } \{0\}\times\Omega\,, \end{aligned}$$

and such that

$$\begin{aligned} &\int_0^T \langle \rho_1\, v_\delta',\, \phi\rangle + \int_{Q_1} \rho_1\,\big((v_\delta)_{(\delta)}\cdot\nabla\big)v_\delta\cdot\phi + \int_{Q_1}\eta(\theta_\delta)\,D(v_\delta\,,\,\phi) \\ &\qquad = \int_{Q_1}\big(\operatorname{curl} H_\delta\times[\mu\, H_\delta]_{(\delta)}\big)\cdot\phi + \int_{Q_1} f(\theta_\delta)\cdot\phi\,, \end{aligned} \tag{7.6}$$

$$\begin{aligned} &\int_0^T \langle \mu\, H_\delta',\, \psi\rangle + \int_{\tilde{Q}} r(\theta_\delta)\,\operatorname{curl} H_\delta\cdot\operatorname{curl}\psi \\ &\qquad = \int_{Q_1}\big(v_\delta\times[\mu\, H_\delta]_{(\delta)}\big)\cdot\operatorname{curl}\psi + \int_{\tilde{Q}} r(\theta_\delta)\, j_g\cdot\operatorname{curl}\psi\,, \end{aligned} \tag{7.7}$$

$$\begin{aligned} &\int_0^T \langle \rho\, c_V\,\theta_\delta',\xi\,\rangle + \int_{Q_1}\rho_1\, c_V\, v_\delta\cdot\nabla\theta_\delta\,\xi + \int_Q\big(\delta\,|\nabla\theta_\delta|^{p-2} + \kappa(\theta_\delta)\big)\,\nabla\theta_\delta\cdot\nabla\xi + \int_{\mathcal{S}} G(\sigma\,\theta_\delta^4)\,\xi \\ &\qquad = \int_{Q\setminus Q_1}\Big[r(\theta_\delta)\,|\operatorname{curl} H_\delta|^2\Big]_{(\delta)}\,\xi\,, \end{aligned} \tag{7.8}$$

for all $\{\phi,\, \psi,\, \xi\} \in \mathcal{V}_0$. Here, $\langle\cdot,\,\cdot\rangle$ are the duality pairings corresponding to the definition of $\mathcal{V}_0$. We additionally have that $\operatorname{ess\,inf}_Q \theta_\delta \geq \theta_g$.

Proof. For $\{v,\, H,\, \theta\} \in \mathcal{V}_0$, we introduce the notations

$$\hat{v} = v + v_g, \quad \hat{\theta} = \theta + \theta_g\,.$$

We consider the operator $\mathcal{A} : \mathcal{V}_0 \longrightarrow \mathcal{V}_0^*$ given by

$$\begin{aligned}\big\langle \mathcal{A}\big(\{v,\, H,\, \theta\}\big)\,,\, \{\phi,\, \psi,\, \xi\}\,\big\rangle &:= \int_{Q_1} \rho\,\big((\hat v)_{(\delta)}\cdot\nabla\big)\hat v\cdot\phi + \int_{Q_1}\eta(\hat\theta)\,D(\hat v\,,\,\phi)\\
&- \int_{Q_1}\big(\operatorname{curl} H\times[\mu\, H]_{(\delta)}\big)\cdot\phi - \int_{Q_1} f(\hat\theta)\cdot\phi + \int_{\tilde Q} r(\hat\theta)\,\operatorname{curl} H\cdot\operatorname{curl}\psi\\
&- \int_{Q_1}\big(\hat v\times[\mu\, H]_{(\delta)}\big)\cdot\operatorname{curl}\psi - \int_{\tilde Q} r(\hat\theta)\, j_g\cdot\operatorname{curl}\psi + \int_Q \rho\, c_V\,\hat v\cdot\nabla\hat\theta\,\xi\\
&+ \int_Q [\delta\,|\nabla\hat\theta|^{p-2}+\kappa(\hat\theta)]\,\nabla\hat\theta\cdot\nabla\xi + \int_{\mathbb S} G(\sigma\,|\hat\theta|^3\,\hat\theta)\,\xi - \int_{Q\setminus Q_1}\Big[r(\hat\theta)\,|\operatorname{curl} H|^2\Big]_{(\delta)}\xi\,.\end{aligned}$$

We easily verify that $\mathcal{A}$ is well defined. We now prove that $\mathcal{A}$ is coercive and pseudomonotone with respect to $D(L)$. Observe that

$$\int_{Q_1}\rho_1\,\big((\hat v)_{(\delta)}\cdot\nabla\big)v\cdot v = \int_{Q_1}\rho_1\,(\hat v)_{(\delta)}\cdot\frac{1}{2}\,\nabla|v|^2 = 0\,,$$

since $(\hat v)_{(\delta)}$ is divergence free and $v = 0$ on $]0,T[\times\partial\Omega_1$. Using integration by parts, we then have

$$\left|\int_{Q_1}\rho_1\,\big((\hat v)_{(\delta)}\cdot\nabla\big)\hat v\cdot v\right| = \left|-\int_{Q_1}\rho_1\,\big((\hat v)_{(\delta)}\cdot\nabla\big)v\cdot v_g\right| \le c_\delta\,\|v_g\|_{L^2(Q_1)}\,\|\nabla v\|_{L^2(Q_1)}\,. \tag{7.9}$$

Considering the fact that $\big(\operatorname{curl} H\times[\mu\, H]_{(\delta)}\big)\cdot v = -\big(v\times[\mu\, H]_{(\delta)}\big)\cdot\operatorname{curl} H$, and the estimate

$$\left|\int_{Q_1}\big(v_g\times[\mu\, H]_{(\delta)}\big)\cdot\operatorname{curl} H\right| \le \frac{2}{\delta}\,\|v_g\|_{L^2(Q_1)}\,\|\operatorname{curl} H\|_{L^2(Q_1)}\,,$$

we find that

$$\left|\int_{Q_1}\big(\operatorname{curl} H\times[\mu\, H]_{(\delta)}\big)\cdot v + \int_{Q_1}\big(\hat v\times[\mu\, H]_{(\delta)}\big)\cdot\operatorname{curl} H\right| \le \frac{2}{\delta}\,\|v_g\|_{L^2(Q_1)}\,\|\operatorname{curl} H\|_{L^2(Q_1)}\,. \tag{7.10}$$

On the other hand, by Young's inequality, we find for all $\gamma > 0$ a constant C_γ such that

$$\left|\int_{Q_1} f(\hat\theta)\cdot v\right| \le \|v\|_{L^2(Q_1)}\,\|f(\hat\theta)\|_{L^2(Q_1)} \le \gamma\,\Big(\|\theta\|^p_{L^p(Q_1)} + \|v\|^2_{L^2(Q_1)}\Big) + C_\gamma\,, \tag{7.11}$$

whenever the growth of f at infinity is less than $p/2$. This is naturally satisfied by f with (7.2). Observe that

$$\int_Q \hat v\cdot\nabla\theta\,\theta = \int_Q \hat v\cdot\nabla\left(\frac{1}{2}\,\theta^2\right) = 0\,,$$

In view of the assumption (7.4), the elementary properties of the operator G give that

$$\int_{\mathbb S} G(\sigma\,|\hat\theta|^3\,\hat\theta)\,\theta = \int_{\mathbb S} G(\sigma\,|\hat\theta|^3\,\hat\theta)\,(\hat\theta-\theta_g) \ge 0\,. \tag{7.12}$$

Summing up the last estimates, we obtain that

$$\begin{aligned}&\big\langle \mathcal{A}\big(\{v,H,\theta\}\big)\,,\,\{v,H,\theta\}\,\big\rangle\\&\quad\geq \int_{Q_1}\eta(\hat\theta)\,D(\hat v\,,v)+\int_{\tilde Q}r(\hat\theta)\,|\operatorname{curl}H|^2+\int_Q\Big(\delta\,|\nabla\hat\theta|^{p-2}\,+\kappa(\hat\theta)\Big)\,\nabla\hat\theta\cdot\nabla\theta\\&\quad-\gamma\,\Big(\|\nabla v\|^2_{L^2(Q_1)}+\|\operatorname{curl}H\|^2_{L^2(Q_1)}+\|\nabla\theta\|^p_{L^p(Q_1)}\Big)-C_{\gamma,\delta}\,.\end{aligned}$$

The coercivity of $\mathcal{A}$ follows easily from a suitable choice of γ by means of Young's inequality.

In order to prove that $\mathcal{A}$ is pseudomonotone, we consider a sequence

$$\{v_k\,,H_k\,,\theta_k\}\rightharpoonup\{v\,,H\,,\theta\}\quad\text{ in } D(L)\,,\tag{7.13}$$

such that

$$\limsup_{k\to\infty}\,\big\langle \mathcal{A}\big(\{v_k,\,H_k,\,\theta_k\}\big)\,,\,\{v_k-v,\,H_k-H,\,\theta_k-\theta\}\,\big\rangle\leq 0\,.\tag{7.14}$$

By the compactness of the embedding $\mathcal{H}_\mu(\tilde\Omega)\hookrightarrow[L^2(\tilde\Omega)]^3$ (see Lemma A.4.1) and the compactness results of Lions [1969], Ch. 1, Th. 5.1, the property (7.13) implies the existence of a subsequence, that we not relabel, for which

$$v_k\longrightarrow v\ \text{ in } [L^2(Q_1)]^3,\quad H_k\longrightarrow H\ \text{ in } [L^2(\tilde Q)]^3,\quad \theta_k\longrightarrow\theta\ \text{ in } L^p(Q)\,.\tag{7.15}$$

Since for all $\gamma>0$ we have the inequality $\|\theta_k-\theta\|_{L^p(\mathcal{S})}\leq\gamma\,\|\nabla(\theta_k-\theta)\|_{L^p(Q)}+c_\gamma\,\|\theta_k-\theta\|_{L^p(Q)}$, we find with the help of (7.15) a subsequence such that

$$\theta_k\longrightarrow\theta\quad\text{ in } L^p(\mathcal{S})\,.\tag{7.16}$$

In view of (7.15), (7.16) and property (7.14), standard rearrangements of terms yield

$$\limsup_{k\to\infty}\Big(\int_{Q_1}D(v_k-v\,,\,v_k-v)+\int_{\tilde Q}|\operatorname{curl}(H_k-H)|^2+\int_Q|\nabla(\theta_k-\theta)|^2\Big)\leq 0\,.$$

This proves that

$$|\operatorname{curl}H_k|^2\longrightarrow|\operatorname{curl}H|^2\ \text{ in } L^1(\tilde Q)\,,\tag{7.17}$$

and implies that $[r(\hat\theta_k)\,|\operatorname{curl}H_k|^2]_{(\delta)}\longrightarrow[r(\hat\theta)\,|\operatorname{curl}H|^2]_{(\delta)}$ in $L^1(Q)$. We are now able to prove that

$$\begin{aligned}&\liminf_{k\to\infty}\,\big\langle \mathcal{A}\big(\{v_k,\,H_k,\,\theta_k\}\big)\,,\,\{v_k-\phi,\,H_k-\psi,\,\theta_k-\xi\}\,\big\rangle\\&\qquad\geq\big\langle \mathcal{A}\big(\{v,\,H,\,\theta\}\big)\,,\,\{v-\phi,\,H-\psi,\,\theta-\xi\}\,\big\rangle\,,\end{aligned}$$

for all $\{\phi,\,\psi,\,\xi\}\in\mathcal{V}_0$. The arguments that lead to this assertion being fairly standard, we do not execute this part of the proof in greater details. Consider now the linear continuous functional $\mathcal{F}\in\mathcal{V}_0^*$ given by

$$\big\langle\mathcal{F},\,\{\phi,\,\psi,\,\xi\}\,\big\rangle:=-\int_{Q_1}\rho_1\,\frac{\partial v_g}{\partial t}\cdot\phi\,.$$

By the properties of the operators L and $\mathcal{A}$, the result of Lions [1969], Ch. 3, Th. 1.2 shows that the evolution problem $(L+A)u = F$ has at least one solution in $u \in \mathcal{V}_0$. This proves the existence of a triple $\{v, H, \theta\} \in \mathcal{V}_0$ such that $\{\hat{v}, H, \hat{\theta}\}$ satisfies the integral relations (7.6), (7.7), and (7.8). From now on, we write again v, H, θ instead of $\hat{v}$, H, $\hat{\theta}$.

In the relation (7.8), we are allowed to choose $\xi = (\theta - \theta_g)^-$. We easily obtain that $\int_Q |\nabla(\theta-\theta_g)^-|^2 \leq 0$. Thus, we can replace $|\theta|^3\,\theta$ by θ^4 on $\mathcal{S}$, which proves the last claim and the proposition. □

For stating the next proposition, we introduce the abbreviation

$$K_g := \|v_g\|_{[V_2^1(Q_1)]^3} + \|j_g\|_{[L^2(\tilde{Q}_{c_0})]^3}\,. \tag{7.18}$$

In the next proposition, we obtain uniform estimates for our approximating sequence.

Proposition 7.1.3. Let the assumptions of Theorem 6.1.1 be satisfied. Let the numbers K_g be given by (7.18). We consider any sequence $\{v_\delta, H_\delta, \theta_\delta\}$ constructed as in Proposition 7.1.2.

(1) We can find a constant $C > 0$ independent of δ such that

$$\|v_\delta\|_{[V_2^{1,0}(Q_1)]^3} + \|H_\delta\|_{[L^{\infty,2}(\tilde{Q})]^3} + \|H_\delta\|_{\mathcal{H}_\mu(\tilde{Q})} \leq C\left(K_g + \|f(\theta_\delta)\|_{[L^2(Q_1)]^3}\right).$$

(2) We also find that

$$\|\theta_\delta - \theta_g\|_{L^{\infty,1}(Q)} \leq C\left(K_g^2 + \|f(\theta_\delta)\|^2_{[L^2(Q_1)]^3}\right),$$

with a positive constant C independent of δ.

(3) For all $1 \leq r < \frac{5}{4}$, we can find a continuous function $\mathcal{P}_r$ that depends on r such that

$$\|\nabla\theta_\delta\|_{[L^r(Q)]^3} \leq \mathcal{P}_r\Big(K_g,\ \|f(\theta_\delta)\|_{[L^2(Q_1)]^3}\Big).$$

Proof. For the sake of notational simplicity, we write v instead of v_δ, etc.

(1): We start from relation (7.6). We can show that for almost all $t \in]0,T[$, and for all $\phi \in D_0^{1,2}(\Omega_1)$, we have

$$\begin{aligned}\big\langle \rho_1\, v'(t)\,,\, \phi\big\rangle + \int_{\Omega_1} \rho_1\left(\left(v(t)\right)_{(\delta)}\cdot\nabla\right)v(t)\cdot\phi + \int_{\Omega_1} \eta(\theta(t))\, D(v(t)\,,\,\phi)\\ = \int_{\Omega_1}\left(\operatorname{curl} H(t)\times[\mu\, H(t)]_{(\delta)}\right)\cdot\phi + \int_{\Omega_1} f(\theta(t))\cdot\phi\,.\end{aligned} \tag{7.19}$$

By classical arguments, we can show that the subset of $]0,T[$ of zero Lebesgue measure where this relation fails can be chosen independently of ϕ and δ. Analogously, from relation (7.7), it follows for almost all $t \in]0,T[$ and all $\psi \in \mathcal{H}_\mu(\tilde{\Omega})$, that

$$\big\langle \mu\, H'(t)\,,\, \psi\big\rangle + \int_{\tilde{\Omega}} r(\theta(t))\ \operatorname{curl} H(t)\cdot\operatorname{curl}\psi \tag{7.20}$$

$$= \int_{\Omega_1}\left(v(t)\times[\mu\, H(t)]_{(\delta)}\right)\cdot\operatorname{curl}\psi + \int_{\tilde{\Omega}} r(\theta(t))\, j_g(t)\cdot\operatorname{curl}\psi\,. \tag{7.21}$$

We can insert the vector field $\phi = v(t) - v_g(t)$ in the relation (7.19) and $\psi = H(t)$ in (7.20). Integrating over the interval $]0, t[$, we obtain on the one hand

$$\begin{aligned}
&\frac{\rho_1}{2}\,\|v(t)\|^2_{L^2(\Omega_1)} + \int_{Q_{1,t}} \eta(\theta)\, D(v,v) \\
&\quad = \frac{\rho_1}{2}\,\|v_0\|^2_{L^2(\Omega_1)} + \int_{\Omega_1} \rho_1\, v(t)\cdot v_g(t) - \int_{\Omega_1} \rho_1\, v_0 \cdot v_g(0) - \int_{Q_{1,t}} \rho_1\, v \cdot \frac{\partial v_g}{\partial t} \\
&\quad + \int_{Q_{1,t}} \big((v)_{(\delta)}\cdot\nabla\big)v\cdot v_g + \int_{Q_1} \big(\operatorname{curl} H \times [\mu\, H]_{(\delta)}\big)\cdot(v - v_g) + \int_{Q_1} f(\theta)\cdot(v - v_g)\,,
\end{aligned} \tag{7.22}$$

where we performed an integration by parts with respect to time. Recalling that $H(0) = 0$, we obtain on the other hand that

$$\int_{\tilde\Omega} \frac{\mu}{2}\,|H(t)|^2 + \int_{\tilde Q_t} |\operatorname{curl} H|^2 = \int_{Q_{1_t}} \big(v\times[\mu\, H]_{(\delta)}\big)\cdot \operatorname{curl} H + \int_{\tilde Q_t} r(\theta) j_g \cdot \operatorname{curl} H\,. \tag{7.23}$$

We add (7.22) and (7.23). For almost all $t < T$, it follows that

$$\begin{aligned}
&\frac{\rho_1}{2}\,\|v(t)\|^2_{L^2(\Omega_1)} + \int_{\tilde\Omega} \frac{\mu}{2}\,|H(t)|^2 + \int_{Q_{1,t}} \eta(\theta)\, D(v,v) + \int_{\tilde Q_t} r(\theta)\,|\operatorname{curl} H|^2 \\
&\quad = \int_{Q_{1,t}} \rho_1\,\big((v)_{(\delta)}\cdot\nabla\big)v\cdot v_g + \int_{\Omega_1} \rho_1\, v(t)\cdot v_g(t) - \frac{\rho_1}{2}\,\|v_0\|^2_{L^2(\Omega_1)} - \int_{Q_{1,t}} \rho_1\, v\cdot\frac{\partial v_g}{\partial t} \\
&\qquad - \int_{Q_1} \big(\operatorname{curl} H \times [\mu\, H]_{(\delta)}\big)\cdot v_g + \int_{Q_{1,t}} f(\theta)\cdot(v - v_g) - \int_{\tilde Q} r(\theta)\, j_g \cdot \operatorname{curl} H\,.
\end{aligned}$$

Using Young's inequality, we produce the estimate

$$\begin{aligned}
\left|\int_{Q_{1,t}} \big((v)_{(\delta)}\cdot\nabla\big)v\cdot v_g\right| &\le \|v_g\|_{L^\infty(Q_1)}\,\|v\|_{L^2(Q_{1,t})}\,\|\nabla v\|_{L^2(Q_{1,t})} \\
&\le \gamma\,\|\nabla v\|^2_{L^2(Q_{1,t})} + \frac{1}{4\,\gamma}\,\|v_g\|^2_{L^\infty(Q_1)}\,\|v\|^2_{L^2(Q_{1,t})}\,.
\end{aligned}$$

We get similarly

$$\left|\int_{Q_1} \big(\operatorname{curl} H\times[\mu\, H]_{(\delta)}\big)\cdot v_g\right| \le \gamma\,\|\operatorname{curl} H\|^2_{L^2(Q_{1,t})} + \frac{1}{4\,\gamma}\,\|v_g\|^2_{L^\infty(Q_1)}\,\|\mu\, H\|^2_{L^2(Q_{1,t})}\,.$$

Further, we can use Poincaré's inequality to obtain that

$$\begin{aligned}
\left|\int_{Q_{1,t}} f(\theta)\cdot(v - v_g)\right| &\le \int_0^t \|f(\theta)\|_{L^2(\Omega_1)}\,\|v - v_g\|_{L^2(\Omega_1)} \\
&\le \gamma\,\|\nabla(v - v_g)\|^2_{L^2(\Omega_1)} + \frac{\operatorname{diam}(\Omega_1)^2}{4\,\gamma}\,\|f(\theta)\|^2_{L^2(Q_{1,t})}\,.
\end{aligned}$$

In view of the assumption (3.44), we therefore obtain the inequality

$$\|v(t)\|^2_{L^2(\Omega_1)} + \int_{\tilde\Omega} |H(t)|^2 + \int_{Q_{1,t}} D(v,v) + \int_{\tilde Q_t} |\operatorname{curl} H|^2$$
$$\leq c \left(K_g^2 + \|f(\theta)\|^2_{L^2(Q_{1,t})} + \|v\|^2_{L^2(Q_{1,t})} + \|H\|^2_{L^2(Q_{1,t})} \right) .$$

Here, the number K_g is given by (7.18), and the constant c depends on the lower and upper bounds (3.44) of the coefficients. Gronwall's Lemma gives the estimate

$$\operatorname*{ess\,sup}_{t\in]0,T[} \left\{ \|v(t)\|^2_{L^2(Q_1)} + \|H(t)\|^2_{L^2(\tilde Q)} \right\} \leq C \left(K_g^2 + \|f(\theta)\|^2_{L^2(Q_1)} \right) , \tag{7.24}$$

which, in turn, implies that

$$\int_{Q_1} D(v,v) + \int_{\tilde Q} |\operatorname{curl} H|^2 \leq \tilde C \left(K_g^2 + \|f(\theta)\|^2_{L^2(Q_1)} \right) . \tag{7.25}$$

This proves (1).

Next, we want to obtain uniform estimates on the temperature. The arguments we use here are completely similar to the ones of the paper Druet [2008a], so we only sketch the proof.

Consider an arbitrary absolutely continuous function $g : \mathbb{R} \to \mathbb{R}$ which is globally bounded. We denote by F the primitive function of g that vanishes at zero. Using Lemma A.3.1, we can prove for all $t_1 < T$ the relation

$$\int_\Omega \rho\, c_V\, F\big(\theta(t_1) - \theta_g\big) + \int_{Q_{1,t_1}} \rho_1\, c_{V1}\, v \cdot \nabla\theta\, g(\theta - \theta_g)$$
$$+ \int_{Q_{t_1}} \big(\delta\, |\nabla\theta|^{p-2} + \kappa(\theta)\big)\, \nabla\theta \cdot \nabla g(\theta - \theta_g) + \int_{\mathcal{S}_{t_1}} G(\sigma\, \theta^4)\, g(\theta - \theta_g)$$
$$= \int_{Q_{t_1}\setminus Q_{1,t_1}} \Big[r(\theta)\, |\operatorname{curl} H|^2 \Big]_{(\delta)}\, g(\theta - \theta_g) \, . \tag{7.26}$$

(2): We observe that the properties of v imply that

$$\int_{Q_{1,t_1}} \rho_1\, c_{V1}\, v \cdot \nabla\theta\, g(\theta - \theta_g) = \int_{Q_{1,t_1}} \rho_1\, c_{V1}\, v \cdot \nabla F(\theta - \theta_g) = 0 \, .$$

If we in addition assume that the function g is monotonely increasing, then

$$\int_{Q_{t_1}} \big(\delta\, |\nabla\theta|^{p-2} + \kappa(\theta)\big)\, \nabla\theta \cdot \nabla g(\theta - \theta_g) = \int_{Q_{t_1}} \big(\delta\, |\nabla\theta|^{p-2} + \kappa(\theta)\big)\, |\nabla\theta|^2\, g'(\theta - \theta_g) \geq 0 \, .$$

For each monotonely increasing, globally Lipschitz continuous and bounded real-valued function g, it follows from (7.26) that

$$\int_\Omega \rho\, c_V\, F\big(\theta(t_1) - \theta_g\big) + \int_{\mathcal{S}_{t_1}} G(\sigma\, \theta^4)\, g\big(\theta - \theta_g\big) \leq \int_{Q_{t_1}\setminus Q_{1,t_1}} \Big[r(\theta)\, |\operatorname{curl} H|^2 \Big]_{(\delta)}\, g\big(\theta - \theta_g\big) \, , \tag{7.27}$$

where F denotes the primitive function of g that vanishes at zero. In (7.27), we choose

$$g(t) = g_\gamma(t) = \frac{1}{\gamma}\,\mathrm{sign}(t)\ \min\{|t|\,,\,\gamma\}\,,$$

as test function, and want to pass to the limit $\gamma \to 0$. We have by Lemma 4.3.6 that

$$\int_{\mathcal{S}_{t_1}} G(\sigma\,\theta^4)\,g\big(\theta - \theta_g\big) \geq 0\,.$$

We use the fact that $g_\gamma \leq 1$ globally, and we use (1) for estimating the right-hand side. In the limit $\gamma \to 0$, we therefore obtain the inequality

$$\int_\Omega \rho\,c_V\,|\theta(t_1) - \theta_g| \leq c\,\left(K_g^2 + \|f(\theta)\|^2_{L^2(Q_1)}\right)\,,$$

which is nothing but (2).

Finally, we prove (3). Let $\gamma \in]0,1[$ be a parameter that we will fix later. In (7.26), we now consider the test function

$$g(s) = g_\gamma\big(s\big) := \mathrm{sign}\big(s\big)\,\left(1 - \frac{1}{(1+|s|)^\gamma}\right)\,.$$

If F denotes the primitive function of g that vanishes at zero, then

$$F(s) = \frac{1}{1-\gamma} + |s| - \frac{(1+|s|)^{1-\gamma}}{1-\gamma} \geq 0 \quad \text{ on } \mathbb{R}\,. \tag{7.28}$$

It follows from Lemma 4.3.6 that

$$\int_{\mathcal{S}_{t_1}} G(\sigma\,\theta^4)\,g_\gamma(\theta - \theta_g) \geq 0\,.$$

On the other hand, since $g_\gamma \leq 1$ globally, we can reuse the estimate (1). the relation (7.26) implies the inequality

$$\begin{aligned}\int_\Omega \rho\,c_V\,F\big(\theta(t_1) - \theta_g\big) + \int_{Q_{t_1}} \frac{\delta\,\gamma\,|\nabla\theta|^p}{(1+|\theta-\theta_g|)^{1+\gamma}} + \int_{Q_{t_1}} \frac{\kappa(\theta)\,\gamma\,|\nabla\theta|^2}{(1+|\theta-\theta_g|)^{1+\gamma}}\\ \leq c\,(K_g^2 + \|f(\theta)\|^2_{L^2(Q_1)})\,.\end{aligned}$$

Since F is globally positive, we obtain for $\gamma \in]0,1[$ arbitrary that

$$\int_{Q_{t_1}} \kappa(\theta)\,\frac{\gamma}{2}\,\frac{|\nabla\theta|^2}{(1+|\theta-\theta_g|)^{1+\gamma}} \leq c\,\left(K_g^2 + \|f(\theta)\|^2_{L^2(Q)}\right)\,.$$

We apply Lemma A.2.5 in order to prove (3). This finishes the proof of the propostion. □

In order to pass to the limit with the approximate solutions, we still need informations concerning the time-derivatives of the approximations. Since this part of the proof is crudely technical, we give it in Lemma 7.1.5 at the end of the section.

Proposition 7.1.4. We consider any sequence $\{v_\delta,\ H_\delta,\ \theta_\delta\}$ of approximate solutions of Proposition 7.1.2. Then there is a subsequence such that

$$\begin{aligned} v_\delta &\rightharpoonup v \ \text{ in } D_2^{1,0}(Q_1), \quad v_\delta \longrightarrow v \ \text{ in } [L^2(Q_1)]^3\,, \\ H_\delta &\rightharpoonup \ H \text{ in } \mathcal{H}_\mu(\tilde{Q}), \quad H_\delta \longrightarrow H \ \text{ in } [L^2(\tilde{Q})]^3\,, \\ \theta_\delta &\rightharpoonup \theta \ \text{ in } W_r^{1,0}(Q)\,, \quad \theta_\delta \longrightarrow \theta \ \text{ in } L^r(Q)\,, \end{aligned}$$

for all $1 \le r < 5/4$. In addition, we have:

(1) There are a subsequence and a Borel regular, nonnegative Radon measure $\nu \in \mathcal{M}(\overline{Q})$ such that

$$\nu_{|_{Q_c\setminus\overline{Q_1}}} = r(\theta)\,|\operatorname{curl} H|^2\, d\,\lambda_4, \quad \nu_{|_{(\overline{Q}\setminus Q_c)\cup\overline{Q_1}}} = 0\,,$$

and such that $\Big[r(\theta_\delta)\,|\operatorname{curl} H_\delta|^2\Big]_{(\delta)} \chi_{Q_c\setminus Q_1} \rightharpoonup \nu$ weakly as measures in $\overline{Q}$.

(2) There exists a functional $\mathcal{G} = \mathcal{G}(\theta) \in [W^1_{q,\mathcal{C}}(Q)]^*$ $(q > 5)$ such that

$$\langle \mathcal{G}, \xi\rangle = \lim_{\delta\to 0} \int_{\mathcal{S}} G(\sigma\,\theta_\delta^4)\,\xi, \quad \xi \in W^1_{q,\mathcal{C}}(Q)\,.$$

If $\xi \in W^1_{q,\mathcal{C}}(Q)$ is nonnegative in Q, then

$$-\langle \mathcal{G}, \xi\rangle + \int_{\mathcal{S}} \sigma\,\theta^4\,\xi \ge \int_{\mathcal{S}} \sigma\,\theta^4\,\mathbf{H}(\xi)\,.$$

(3) There exists a Borel regular, signed Radon measure $\tilde{\nu} \in \mathcal{M}(Q)$ such that the measure $\nu - \tilde{\nu}$ is positive in Q and

$$\langle \mathcal{G}, \xi\rangle = \int_{\mathcal{S}} \sigma\,\theta^4\,G(\xi) + \int_{[0,T]\times\partial\Omega_c} \xi\, d\tilde{\nu},$$

for all $\xi \in C^\infty(0,T; C^\infty_\Gamma(\Omega))$ such that $\xi(T) = 0$ that satisfy (3.76).

Proof. The weak convergence of $\{v_\delta\}$ in the space $D_2^{1,0}(Q_1)$ and the strong convergence in $[L^2(Q_1)]^3$ are direct consequences of Proposition 7.1.3, of Lemma 7.1.5 below, and of the compactness result of Lions [1969], Ch. 1, Th. 5.1. Similarly, we obtain the weak convergence of $\{H_\delta\}$ in $\mathcal{H}_\mu(\tilde{Q})$ from the Hilbert space structure of this space and the bounds of Proposition 7.1.3. We obtain the strong convergence in $[L^2(\tilde{Q})]^3$ from the compactness assertion of Lemma A.4.1 and from Lions [1969]. In view of Proposition 7.1.3, we will also find a subsequence such that for all $1 \le r < \frac{5}{4}$,

$$\theta_\delta \rightharpoonup \theta \quad \text{in } W_r^{1,0}(Q)\,. \tag{7.29}$$

If s is given as in Lemma 7.1.5, (2), we have the well-known situation $W^{1,r}(\Omega_i) \hookrightarrow L^r(\Omega_i) \hookrightarrow [W_0^{1,s'}(\Omega_i)]^*$, the first of these injections being compact. From Lemma 7.1.5, and the generalized Lemma of Aubin-Lions (see Simon [1986]), we get for all $i = 0, \dots, m$

the existence of a subsequence, that we not relabel, such that $\theta_\delta \longrightarrow \theta$ in $L^r(Q_i)$. In view of the inequality

$$\|\theta_\delta - \theta\|_{L^r(]0,T[\times\partial\Omega_i)} \leq \gamma\,\|\theta_\delta - \theta\|_{W^{1,0}_r(Q_i)} + c_\gamma\,\|\theta_\delta - \theta\|_{L^r(Q_i)}\,,$$

which is valid for all $\gamma > 0$, we have $\limsup_{\delta\to 0}\,\|\theta_\delta - \theta\|_{L^r(]0,T[\times\partial\Omega_i)} \leq \gamma\,C$. Therefore, we can choose a subsequence such that $\theta_\delta \longrightarrow \theta$ in $L^r(]0,T[\times\partial\Omega_i)$, and after extracting subsequences even

$$\theta_\delta \longrightarrow \theta \text{ in } L^r(Q)\,, \quad \theta_\delta \longrightarrow \theta \text{ in } L^r(\mathcal{S})\,, \quad \theta_\delta \longrightarrow \theta \text{ pointwise a. e. in } Q \text{ and on } \mathcal{S}\,. \tag{7.30}$$

We now discuss the additional convergence assumptions.

(1): Since the sequence $\left\{|\operatorname{curl} H_\delta|^2\,\chi_{Q_c\setminus Q_1}\right\}$ is bounded in the space $L^1(Q)$, we immediately find a Borel-regular Radon measure ν such that

$$\Big[r(\theta_\delta)\,|\operatorname{curl} H_\delta|^2\Big]_{(\delta)}\,\chi_{Q_c\setminus Q_1} \rightharpoonup \nu \qquad \text{weakly as measures in } \overline{Q}\,. \tag{7.31}$$

The measure ν is obviously positive. We now want to prove that for all $\hat\Omega \subset\subset \Omega_c \setminus \Omega_1$,

$$\Big[r(\theta_\delta)\,|\operatorname{curl} H_\delta|^2\Big]_{(\delta)} \longrightarrow r(\theta)\,|\operatorname{curl} H|^2 \text{ in } L^1(]0,T[\times\hat\Omega)\,. \tag{7.32}$$

First considering an arbitrary $\psi \in L^2(0,T;\mathcal{H}_\mu(\tilde\Omega))$ supported in $\tilde Q \setminus Q_1$, we can pass to the limit in (7.8) and obtain the relation

$$\int_0^T \langle \mu\, H'\,,\,\psi\rangle + \int_{\tilde Q} r(\theta)\,\operatorname{curl} H\cdot\operatorname{curl}\psi = \int_{\tilde Q} r(\theta)\,j_g\cdot\operatorname{curl}\psi\,,$$

For $0 < \delta_1$ arbitrary and $\psi \in L^2(0,T;\mathcal{H}_\mu(\tilde\Omega))$ supported in $\tilde Q \setminus Q_1$, we thus have the relation

$$\begin{aligned}\int_0^T \langle \mu\,(H_{\delta_1} - H)'\,,\,\psi\rangle &+ \int_{\tilde Q} r(\theta_{\delta_1})\,\operatorname{curl}(H_{\delta_1} - H)\cdot\operatorname{curl}\psi \\ &= \int_{\tilde Q} [r(\theta) - r(\theta_{\delta_1})]\,(\operatorname{curl} H - j_g)\cdot\operatorname{curl}\psi\,.\end{aligned} \tag{7.33}$$

By Lemma A.4.3, this relation holds even for all $\psi \in \mathcal{H}(\tilde Q)$ that are supported in $\tilde Q \setminus Q_1$. We now consider an arbitrary domain $\hat\Omega \subset\subset \Omega_c\setminus\Omega_1$, and we fix a function $\zeta \in C^\infty_c(\Omega_c\setminus\Omega_1)$, such that $\zeta \equiv 1$ on $\hat\Omega$. The vector field $\psi := (H_{\delta_1} - H)\,\zeta$ obviously belongs to $\mathcal{H}(\tilde Q)$, and is supported in $\tilde Q \setminus Q_1$. We insert this ψ in (7.33), and obtain that

$$\begin{aligned}&\int_{\tilde\Omega} \frac{\mu}{2}\,\zeta\,|H_{\delta_1}(t) - H(t)|^2 + \int_{\tilde Q_t} r(\theta_{\delta_1})\,\zeta\,|\operatorname{curl}(H_{\delta_1} - H)|^2 = \\ &\quad - \int_{\tilde Q_t} r(\theta_{\delta_1})\,\operatorname{curl}(H_{\delta_1} - H)\cdot\big((H_{\delta_1} - H)\times\nabla\zeta\big) \\ &\quad + \int_{\tilde Q_t} [r(\theta) - r(\theta_{\delta_1})]\,(\operatorname{curl} H - j_g)\cdot\operatorname{curl}\big((H_{\delta_1} - H)\,\zeta\big)\,.\end{aligned}$$

Since as $\delta_1 \to 0$

$$(H_{\delta_1} - H) \times \nabla\zeta \longrightarrow 0 \ \text{ in } [L^2(\tilde{Q})]^3 , \qquad [r(\theta) - r(\theta_{\delta_1})] \operatorname{curl} H \longrightarrow 0 \ \text{ in } [L^2(\tilde{Q})]^3 ,$$

we see that

$$r_l \int_{]0,T[\times\hat{\Omega}} |\operatorname{curl}(H_{\delta_1} - H)|^2 \le \int_{\tilde{Q}} r(\theta_{\delta_1})\, \zeta \,|\operatorname{curl}(H_{\delta_1} - H)|^2 \longrightarrow 0 ,$$

proving (7.32). By a classical diagonalization argument, we can find a subsequence such that for all $\hat{\Omega} \subset\subset \Omega_c \setminus \Omega_1$,

$$|\operatorname{curl} H_\delta|^2 \longrightarrow |\operatorname{curl} H|^2 \text{ in } L^1(]0,T[\times\hat{\Omega}) . \tag{7.34}$$

It therefore follows for all $\hat{\Omega} \subset\subset \Omega_c \setminus \Omega_1$ that $[r(\theta_\delta)\,|\operatorname{curl} H_\delta|^2]_{(\delta)} \to r(\theta)\,|\operatorname{curl} H|^2$ in $L^1(]0,T[\times\hat{\Omega})$. Comparing with (7.31), we see that (1) is valid.

(2) and (3): We start from the relation (7.8) and we observe that $\int_{Q_1} \rho_1\, c_{V1}\, v_\delta \cdot \nabla\theta_\delta\, \xi = -\int_{Q_1} \rho_1\, c_{V1}\, \theta_\delta\, v_\delta \cdot \nabla\xi$. We pass to the limit $\delta \to 0$ in (7.8) for arbitrary $\xi \in W^1_{q,\mathcal{C}}(Q)$ $(q > 5)$ such that $\xi(T) = 0$. Considering also the estimate (7.44), we see that $\lim_{\delta\to 0} \int_{\mathcal{S}} G(\sigma\, \theta_\delta^4)\, \xi$ exists, and we obtain the equation

$$\begin{aligned} -\int_Q \rho\, c_V\, \theta \frac{\partial\xi}{\partial t} - \int_{Q_1} \rho_1\, c_{V1}\, \theta\, v \cdot \nabla\xi + \int_Q \kappa(\theta)\, \nabla\theta \cdot \nabla\xi + \lim_{\delta\to 0} \int_{\mathcal{S}} G(\sigma\, \theta_\delta^4)\, \xi \\ = \int_\Omega \theta_0\, \xi(0) + \int_{Q\setminus Q_1} r(\theta)\,|\operatorname{curl} H|^2\, \xi + \int_{[0,T]\times\partial\Omega_c} \xi\, d\nu , \end{aligned} \tag{7.35}$$

We define $\mathcal{G}(\xi) := \lim_{\delta\to 0} \int_{\mathcal{S}} G(\sigma\, \theta_\delta^4)\, \xi$. Our goal is now to prove the additional properties of $\mathcal{G}$.

We would like to employ the same method as in Proposition 4.4.4, but since the right-hand sides converge only in the sense of measures, we have to face additional difficulties. For $\gamma > 0$, we consider the function

$$g(s) = g_\gamma(s) := \frac{1}{1 + \gamma\, s^4} \quad \text{for } s \in \mathbb{R}^+ ,$$

and we denote by $F = F_\gamma$ the primitive function of g that vanishes at zero. Using the fact that g is decreasing on the positive real axis, we can prove, with the techniques of Proposition 4.4.4, the inequality

$$\begin{aligned} -\int_Q \rho\, c_V\, F(\theta_\delta) \frac{\partial\tilde{\xi}}{\partial t} + \int_{Q_1} \rho_1\, c_{V1}\, v_\delta \cdot \nabla\theta_\delta\, g(\theta_\delta)\, \tilde{\xi} + \int_Q \big(\delta\,|\nabla\theta_\delta|^{p-2} + \kappa(\theta_\delta)\big)\, \nabla\theta_\delta \cdot \nabla\tilde{\xi}\, g(\theta_\delta) \\ + \int_{\mathcal{S}} G(\sigma\,|\theta_\delta|^4)\, \tilde{\xi}\, g(\theta_\delta) \ge \int_\Omega \rho\, c_V\, F(\theta_{g,\delta}(0))\, \tilde{\xi}(0) + \int_{Q\setminus Q_1} \Big[r(\theta_\delta)\,|\operatorname{curl} H_\delta|^2\Big]_{(\delta)} \tilde{\xi}\, g(\theta_\delta) , \end{aligned} \tag{7.36}$$

for all $\tilde{\xi} \in W^1_{\infty,\mathcal{C}}(Q)$, such that $\tilde{\xi}(T) = 0$, and $\tilde{\xi} \geq 0$ in Q. The proof of (7.36) is based on Steklov averaging, and on the use of the test functions $\xi_{\gamma,h} = g_\gamma(\theta_{(h)})\,\tilde{\xi}$. Again arguing as in the proof of Proposition 4.4.4, we can prove that

$$\lim_{\delta\to 0}\int_{\mathcal{S}} G(\sigma\,\theta_\delta^4)\,\tilde{\xi}\,g(\theta_\delta) \leq \int_{\mathcal{S}} \sigma\,\theta^4\,g(\theta)\,\tilde{\xi} - \int_{\mathcal{S}} \sigma\,\theta^4\,\mathbf{H}(\tilde{\xi}\,g(\theta))\,.$$

The proof of this last point relies on the properties of the operator G, on dominated convergence, and on Fatou's Lemma.

Passing to the limit $\delta \to 0$ in (7.36), and using Fatou's Lemma in connection with (7.34), we obtain that

$$\begin{aligned} &-\int_Q \rho\,c_V\,F(\theta)\,\frac{\partial\tilde{\xi}}{\partial t} - \int_{Q_1} \rho_1\,c_{V1}\,\theta\,v\cdot\nabla\tilde{\xi}\,g(\theta) + \int_Q \kappa(\theta)\,\nabla\theta\cdot\nabla\tilde{\xi}\,g(\theta) + \int_{\mathcal{S}} \sigma\,\theta^4\,g(\theta)\,\tilde{\xi} \\ &\quad -\int_{\mathcal{S}} \sigma\,\theta^4\,\mathbf{H}(\tilde{\xi}\,g(\theta)) \;\geq \int_\Omega \rho\,c_V\,F(\theta_0)\,\tilde{\xi}(0) + \int_{Q\setminus Q_1} r(\theta)\,|\operatorname{curl} H|^2\,\tilde{\xi}\,g(\theta)\,. \end{aligned} \tag{7.37}$$

We now recall that $F = F_\gamma$ and $g = g_\gamma$. By the definition of these functions, we see easily that F_γ monotonely increases to the identity, and that g_γ monotonely increases to the constant function 1. By this last property, we can write

$$\int_{\mathcal{S}} \sigma\,\theta^4\,g_\gamma(\theta)\,\tilde{\xi} \longrightarrow \int_{\mathcal{S}} \sigma\,\theta^4\,\tilde{\xi}\,,$$

for $\gamma \to 0$. Arguing as in Proposition 4.4.4, we have also

$$\liminf_{\gamma\to 0}\int_{\mathcal{S}} \sigma\,\theta^4\,\mathbf{H}(\tilde{\xi}\,g_\gamma(\theta)) \geq \int_{\mathcal{S}} \sigma\theta^4\,\mathbf{H}(\tilde{\xi})\,.$$

In the limit of (7.37), we then obtain, for test functions $\tilde{\xi}$ that satisfy, the inequality

$$\begin{aligned} &-\int_Q \rho\,c_V\,\theta\,\frac{\partial\tilde{\xi}}{\partial t} - \int_{Q_1} \rho_1\,c_{V1}\,\theta\,v\cdot\nabla\tilde{\xi} + \int_Q \kappa(\theta)\,\nabla\theta\cdot\nabla\tilde{\xi} + \int_{\mathcal{S}} \sigma\,\theta^4\,\tilde{\xi} \\ &\qquad \geq \int_\Omega \rho_1\,c_V\,\theta_0\,\tilde{\xi}(0) + \int_{Q\setminus Q_1} r(\theta)\,|\operatorname{curl} H|^2\,\tilde{\xi} + \int_{\mathcal{S}} \sigma\,\theta^4\,\mathbf{H}(\tilde{\xi})\,. \end{aligned} \tag{7.38}$$

Comparing (7.35) and (7.38), where we choose $\xi = \tilde{\xi}$, we get (2).

In order to prove (3), we in particular consider in (7.35) and (7.38) test functions $\xi \in W^1_{\infty,\mathcal{C}}(Q)$ such that $\xi \geq 0$ in Q and $\xi(T) = 0$, with the additional property

$$\int_0^T \max_{\overline{\Omega}} |\xi(t)| \int_\Sigma \theta^4(t)\,dt\,. \tag{7.39}$$

For this ξ, we can conclude as in the proof of Proposition 4.4.4 that

$$\lim_{\delta\to 0}\int_{\mathcal{S}} \sigma\,\theta_\delta^4\,G(\xi) = \lim_{\delta\to 0}\int_{\mathcal{S}} G(\sigma\,\theta_\delta^4)\,\xi \leq \int_{\mathcal{S}} \sigma\,\theta^4\,G(\xi) + \int_{]0,T[\times\partial\Omega_c} \xi\,d\nu\,, \tag{7.40}$$

Multiplicating the inequality (7.40) with -1, we obtain, for all $\xi \in W^1_{\infty,\mathcal{C}}(Q)$ with (7.39) such that $\xi \le 0$ in Q and $\xi(T) = 0$, that

$$-\int_{[0,T]\times\partial\Omega_c} \xi \, d\nu + \lim_{\delta\to 0} \int_{\mathcal{S}} \sigma\, \theta_\delta^4\, G(\xi) \ge \int_{\mathcal{S}} \sigma\, \theta^4\, G(\xi)\,. \tag{7.41}$$

Using the construction of Lemma 4.4.5, we now can write for arbitrary $\xi \in W^1_{\infty,\mathcal{C}}(Q)$ with (7.39) such that $\xi \ge 0$ in Q and $\xi(T) = 0$ that

$$\begin{aligned}
\lim_{\delta\to 0} \int_{\mathcal{S}} \sigma\, \theta_\delta^4\, G(\xi) - \int_{[0,T]\times\partial\Omega_c} \xi \, d\nu \le \int_{\mathcal{S}} \sigma\, \theta^4\, G(\xi) &= \int_{\mathcal{S}} \sigma\, \theta^4\, G(\bar{\xi}) \\
&\le \lim_{\delta\to 0} \int_{\mathcal{S}} \sigma\, \theta_\delta^4\, G(\bar{\xi}) - \int_{[0,T]\times\partial\Omega_c} \bar{\xi}\, d\nu \\
&= \lim_{\delta\to 0} \int_{\mathcal{S}} \sigma\, \theta_\delta^4\, G(\xi) - \int_{[0,T]\times\partial\Omega_c} \bar{\xi}\, d\nu\,,
\end{aligned}$$

where we made use of (7.40) and (7.41). This implies that

$$\begin{aligned}
\left| \lim_{\delta\to 0} \int_{\mathcal{S}} \sigma\, \theta_\delta^4\, G(\xi) - \int_{\mathcal{S}} \sigma\, \theta^4\, G(\xi) \right| &\le \max\left\{ \left| \int_{[0,T]\times\partial\Omega_c} \xi\, d\nu \right|, \left| \int_{[0,T]\times\partial\Omega_c} \bar{\xi}\, d\nu \right| \right\} \\
&\le \nu(\overline{Q}) \max_{\overline{Q}} |\xi|\,.
\end{aligned} \tag{7.42}$$

We now observe that the set of all $\xi \in W^1_{\infty,\mathcal{C}}(Q)$, with (7.39), such that $\xi \le 0$ in Q and $\xi(T) = 0$, is a linear subspace of $C(\overline{Q})$.

In view of (7.42), the Hahn-Banach theorem gives the existence of a signed measure $\tilde{\nu} \in \mathcal{M}(\overline{Q})$ such that

$$\lim_{\delta\to 0} \int_{\mathcal{S}} \sigma\, \theta_\delta^4\, G(\xi) - \int_{\mathcal{S}} \sigma\, \theta^4\, G(\xi) = \int_{\overline{Q}} \xi\, d\tilde{\nu}\,,$$

for all $\xi \in W^1_{\infty,\mathcal{C}}(Q)$ with (7.39) such that $\xi = 0$ in $\{T\} \times \Omega$. In view of (7.40), we can choose the measure $\tilde{\nu}$ such that

$$\int_{\overline{Q}} \xi\, d\tilde{\nu} \le \int_{\overline{Q}} \xi\, d\nu\,, \tag{7.43}$$

for all $\xi \in C(\overline{Q})$, $\xi \ge 0$. In addition, the measure $\tilde{\nu}$ is supported in $[0,T] \times (\partial\Omega_c \cap \Sigma)$. $\square$

Proof of Theorem 7.1.1. For $1 \le p < 10/9$, observe that

$$\int_0^T \|v(t)\, \theta(t)\|^p_{L^p(\Omega_1)} \le \|v\|^p_{L^{10/3}(Q_1)}\, \|\theta\|^p_{L^z(Q_1)}\,, \tag{7.44}$$

where $z := 10\,p/(10 - 3\,p)$. The choice of $p < 10/9$, we ensure that $z < 5/3$. Observe now that in view of Lemma A.2.3, the estimate

$$\|\theta\|_{L^z(Q)} \le c\, \|\theta\|_{L^{\infty,1}(Q)}\, \|\nabla\theta\|_{[L^r(Q)]^3}\,,$$

is valid with $r < 5/4$. The estimates of Proposition 7.1.3 and (7.44) therefore give a uniform bound for the sequence $\|v(t)\,\theta(t)\|_{L^p(\Omega_1)}$ in the space $L^p(0,T)$.

In view of Proposition 7.1.4, we are able to choose a subsequence such that $\theta_\delta\, v_\delta \rightharpoonup \theta\, v$ in $[L^p(Q_1)]^3$ as $\delta \to 0$. In the limit of (7.8), we obtain for $\xi \in W^1_{q,\mathcal{C}}(Q)$ $(q > 5)$ such that $\xi(T) = 0$ that

$$- \int_Q \rho\, c_V\, \theta\, \frac{\partial \xi}{\partial t} - \int_{Q_1} \rho_1\, c_{V1}\, \theta\, v \cdot \nabla\xi + \int_Q \kappa(\theta)\, \nabla\theta \cdot \nabla\xi + \langle \mathcal{G},\, \xi \rangle$$
$$= \int_\Omega \rho\, c_V\, \theta_0\, \xi + \int_{Q\setminus Q_1} r(\theta)\, |\operatorname{curl} H|^2\, \xi + \int_{[0,T]\times\partial\Omega_c} \xi\, d\,\nu\,.$$

For $\xi \in C^\infty(0,T; C^\infty_\Gamma(\Omega))$ with (3.76) such that $\xi(T) = 0$, we even obtain that

$$- \int_Q \rho\, c_V\, \theta\, \frac{\partial \xi}{\partial t} - \int_{Q_1} \rho_1\, c_{V1}\, \theta\, v \cdot \nabla\xi + \int_Q \kappa(\theta)\, \nabla\theta \cdot \nabla\xi + \int_{\mathcal{S}} \sigma\, \theta^4\; G(\xi)$$
$$= \int_\Omega \rho\, c_V\, \theta_0\, \xi + \int_{Q\setminus Q_1} r(\theta)\, |\operatorname{curl} H|^2\, \xi + \int_{[0,T]\times\partial\Omega_c} \xi\, d\,(\nu - \tilde{\nu})\,,$$

where the measures ν, $\tilde{\nu}$ are given as in Proposition 7.1.4. Note that for an arbitrary $i \in \{0, \ldots, m\}$ and arbitrary $\xi \in C^\infty(0,T; C^\infty_c(\Omega_i))$, we can recall (7.34) and pass to the limit $\delta \to 0$, in order to obtain

$$- \int_{Q_i} \rho_i\, c_{Vi}\, \theta\, \frac{\partial \xi}{\partial t} - \int_{Q_i\cap Q_1} \rho_1\, c_{V1}\, \theta\, v \cdot \nabla\xi + \int_{Q_i} \kappa(\theta)\, \nabla\theta \cdot \nabla\xi$$
$$= \int_{Q_i} \rho_i\, c_{Vi}\, \theta_0\, \xi(0) + \int_{Q_i\setminus Q_1} r(\theta)\, |\operatorname{curl} H|^2\, \xi\,.$$

Therefore, we see that the balance of energy is satisfied without defect measure in the interior of each subdomain. Passage to the limit in the other relations is an elementary exercise in view of Proposition 7.1.4. The theorem is proved. □

Lemma 7.1.5. Let the hypothesis of Theorem 7.1.1 be satisfied. For any sequence $\{v_\delta,\, H_\delta,\, \theta_\delta\}$ according to Propostion 7.1.2, it holds that:

(1) There exist a number $q \in]p-1\,,\, p[$ and a positive constant C that does not depend on δ, such that $\|\delta^{1/p-1}\,\theta_\delta\|_{W^{1,0}_q(Q)} \leq C$.

(2) For all $1 < s < \min\{10/9,\, q/(p-1)\}$ we have

$$\|v'_\delta\|_{L^{5/4}(0,T;[D^{1,2}_0(\Omega_1)]^*)} + \|H'_\delta\|_{L^{5/4}(0,T;[\mathcal{H}_\mu(\tilde{\Omega})]^*)} + \sum_{i=0}^m \|\theta'_\delta\|_{L^1(0,T;[W^{1,s'}_0(\Omega_i)]^*)} \leq C\,,$$

where $s' = s/(s-1)$ denotes the conjugated exponent to s.

Proof. (1): We employ the same strategy as in Proposition 7.1.3. For a fixed δ, we define $\hat{\theta} := \delta^{\frac{1}{p-1}}\,\theta_\delta$ and $\hat{\theta}_g := \delta^{\frac{1}{p-1}}\,\theta_g$. In the remainder of this proof, we do not indicate the subscripts δ. In view of Proposition 7.1.3, (2), we immediately obtain the bound

$$\operatorname*{ess\,sup}_{t\in]0,T[} \|\hat{\theta}(t) - \hat{\theta}_g\|_{L^1(\Omega)} \le c\,\delta^{\frac{1}{p-1}}\,, \tag{7.45}$$

with a positive constant c that depends only on the data. Using the reasoning of Proposition 7.1.3, it is easy to prove for all $\gamma \in]0,1[$ the relation

$$\int_{Q_{t_1}} \gamma\,\frac{|\nabla\hat{\theta}|^p}{(1+|\hat{\theta}-\hat{\theta}_g|)^{1+\gamma}} \le C\,.$$

with a constant C that depends only on the data.

Using Lemma A.2.5, we find the existence of a $q > p-1$ such that

$$\left\|\nabla(\delta^{\frac{1}{p-1}}\,\theta)\right\|_{L^q(Q)} \le C\,.$$

Now, we are able to prove also (2). We recall that the embedding $V_2^{1,0}(Q) \hookrightarrow L^{10/3}(Q)$ is continuous (see Ladyzenskaja et al. [1968], Chapter II, paragraph 3). Estimating the right-hand side of (7.19), we obtain

$$\begin{aligned}
&|\langle v'(t)\,,\,\phi\rangle| \\
&\quad\le c\left(\|v(t)\|_{L^{10/3}(\Omega_1)}\,\|\nabla v(t)\|_{L^2(\Omega_1)} + \|H(t)\|_{L^{10/3}(\Omega_1)}\,\|\operatorname{curl} H(t)\|_{L^2(\Omega_1)}\right)\|\phi\|_{L^5(\Omega_1)} \\
&\qquad + \eta_u\,\|\nabla v(t)\|_{L^2(\Omega_1)}\,\|\phi\|_{L^2(\Omega_1)} + \|f(\theta(t))\|_{L^2(\Omega_1)}\,\|\phi\|_{L^2(\Omega_1)}\,,
\end{aligned}$$

for all $\phi \in D_0^{1,2}(\Omega_1)$. It follows that

$$\begin{aligned}
\|v'(t)\|_{[D_0^{1,2}(\Omega_1)]^*} \le \bar{c}\,\Big(&\|v(t)\|_{L^{10/3}(\Omega_1)}\,\|\nabla v(t)\|_{L^2(\Omega_1)} + \|H(t)\|_{L^{10/3}(\Omega_1)}\,\|\operatorname{curl} H(t)\|_{L^2(\Omega_1)} \\
&+ \eta_u\,\|\nabla v(t)\|_{L^2(\Omega_1)} + \|f(\theta(t))\|_{L^2(\Omega_1)}\Big)\,.
\end{aligned}$$

Note that the sequence on the right-hand side of this last relation is uniformly bounded in the space $L^{5/4}(0,T)$[1]. We can argue the same with relation (7.20) and get

$$\begin{aligned}
&|\langle H'(t)\,,\,\psi\rangle| \\
&\quad\le c\left(\|v(t)\|_{L^{10/3}(\Omega_1)}\,\|\operatorname{curl} H(t)\|_{L^2(\Omega_1)} + \|H(t)\|_{L^{10/3}(\Omega_1)}\,\|\nabla v(t)\|_{L^2(\Omega_1)}\right)\|\psi\|_{L^5(\Omega_1)} \\
&\qquad + r_u\,(\|\operatorname{curl} H(t)\|_{[L^2(\tilde{\Omega})]^3} + \|j_g(t)\|_{[L^2(\tilde{\Omega})]^3})\,\|\operatorname{curl}\psi\|_{L^2(\tilde{\Omega})}\,,
\end{aligned}$$

[1]We here assume that the sequence $\{H_\delta\}$ can be uniformly bounded in the space $[V_2^{1,0}(Q_1)]^3$. We therefore need to apply the result of Proposition 5.2.3, (1), which means stronger assumptions on the geometry than the ones formulated in Theorem 7.1.1. Under unessential modifications of the estimates in Lemma 7.1.5 (cp. Druet [2009b]), Theorem 7.1.1 is however valid in the form stated.

for all $\psi \in \mathcal{H}_\mu(\tilde{\Omega})$. We obtain that

$$\|H'(t)\|_{[\mathcal{H}_\mu(\tilde{\Omega})]^*} \leq c\Big(\|v(t)\|_{L^{10/3}(\Omega_1)}\,\|\mathrm{curl}H(t)\|_{L^2(\Omega_1)} + \|H(t)\|_{L^{10/3}(\Omega_1)}\,\|\nabla v(t)\|_{L^2(\Omega_1)} \\ + \|\,\mathrm{curl}\,H(t)\|_{[L^2(\tilde{\Omega})]^3} + \|j_g(t)\|_{[L^2(\tilde{\Omega})]^3}\Big)\,.$$

Here also, the sequence on the right-hand side of this last relation is uniformly bounded in the space $L^{5/4}(0,T)$. Finally, we verify that (7.8) implies for almost all $t \in]0,T[$ and for all $\xi \in W^{1,p}_\Gamma(\Omega)$ that

$$\langle \rho\, c_V\, \theta'(t)\,,\xi\rangle - \int_{\Omega_1} \rho_1\, c_{V1}\, v(t)\,\theta(t)\cdot\nabla\xi + \int_\Omega \big(\delta\,|\nabla\theta(t)|^{p-2} + \kappa(\theta(t))\big)\,\nabla\theta(t)\cdot\nabla\xi \\ + \int_\Sigma G(\sigma\,\theta(t)^4)\,\xi = \int_{\Omega\setminus\Omega_1}\Big[r(\theta(t))\,|\,\mathrm{curl}\,H(t)|^2\Big]_{(\delta)}\,\xi\,.$$

Note that we have integrated by parts in the convection term. Choosing especially $\xi \in W^{1,p}_0(\Omega_i)$, for $i \in \{0,\ldots,m\}$ arbitrary, we can write

$$\big\langle \rho\, c_V\, \theta'(t)\,,\xi\big\rangle - \int_{\Omega_1} \rho_1\, c_{V1}\, v(t)\,\theta(t)\cdot\nabla\xi + \int_\Omega \big(\delta\,|\nabla\theta(t)|^{p-2} + \kappa(\theta(t))\big)\,\nabla\theta(t)\cdot\nabla\xi \\ = \int_{\Omega\setminus\Omega_1}\Big[r(\theta(t))\,|\,\mathrm{curl}\,H(t)|^2\Big]_{(\delta)}\,\xi\,,$$

We consider q given as in (1) and we choose $1 < s < \min\left\{\frac{10}{9}\,,\,\frac{q}{p-1}\right\}$. Using the continuity of the embedding $W^{1,s'}(\Omega) \hookrightarrow C(\overline{\Omega})$ for $s' > 3$, we can estimate

$$|\langle\theta'(t)\,,\xi\rangle| \leq c\Big(\|v(t)\,\theta(t)\|_{L^s(\Omega_i)} + \|\nabla(\delta^{\frac{1}{p-1}}\,\theta(t))\|^{p-1}_{L^{s\,(p-1)}(\Omega_i)} + \kappa_u\,\|\nabla\theta(t)\|_{L^s(\Omega_i)} \\ + \|\,\mathrm{curl}\,H(t)\|^2_{L^2(\tilde{\Omega})}\Big)\,\|\nabla\xi\|_{L^{s'}(\Omega_i)}\,.$$

The estimates of Proposition 7.1.3 and (7.44) therefore give a uniform bound for the sequence $\|v(t)\,\theta(t)\|_{L^s(\Omega_1)}$ in the space $L^s(0,T)$. It follows that

$$\|\theta'(t)\|_{[W^{1,s'}_0(\Omega_i)]^*} \leq c\Big(\|v(t)\,\theta(t)\|_{L^s(\Omega_i)} + \|\nabla(\delta^{\frac{1}{p-1}}\,\theta(t))\|^{p-1}_{L^{s\,(p-1)}(\Omega_i)} + \kappa_u\,\|\nabla\theta(t)\|_{L^s(\Omega_i)} \\ + \|\,\mathrm{curl}\,H(t)\|^2_{L^2(\tilde{\Omega})}\Big)\,,$$

with right-hand side uniformely bounded in the space $L^1(0,T)$. This finishes the proof of the lemma. □

7.2 Boussinesq approximation

As we already mentioned, the condition (7.2) does not account for the case that the force term f in the Navier-Stokes equations is given by the Boussinesq model (2.5). An argument in favor of (7.2) is that Boussinesq's approximation is inappropriate if the number $\alpha\,(\theta-\theta_{\text{Ref}})$ exceeds a critical (small) value. However, the approach of the first section would be fully justified only if we could prove *a posteriori* that the weak solution obtained in Theorem 6.1.1 actually satisfies (7.2). In this section, we mention a few results related to the last topic. Since similar arguments were already applied in the section 6.2, the proofs are left as an exercise.

Lemma 7.2.1. Assume that the number M_θ given in (7.2) is such that

$$\beta := \min\left\{\frac{\rho\, c_V}{2},\, \kappa\right\} c - 10^{2/3}\, C\ \text{meas}(Q_1)^{2/5}\rho_1^2\,|\vec{g}|^2\, M_\theta\,\alpha > 0\,,$$

with the constant C from Propostion 7.1.3, (1) and the constant c of the continuous embedding $V_2^{1,0}(Q) \hookrightarrow L^{10/3}(Q)$. Then, the weak solution constructed in Theorem 6.1.1 satisfies

$$\left(\frac{1}{\text{meas}(Q_1)}\int_{Q_1} \alpha^{3/2}\,|\theta-\theta_M|^{3/2}\right)^{2/3} \le \frac{\tilde{C}\,\alpha\,K_g^2}{\text{meas}(Q_1)^{3/5}\,\beta}\,,$$

where the number K_g is given by (7.18).

Proof. The proof relies on estimates in Marcinkiewicz spaces (see Lemma A.2.2). □

Remark 7.2.2. If the number α is sufficiently small, Lemma 7.2.1 shows that

$$\left(\frac{1}{\text{meas}(Q_1)}\int_{Q_1} \alpha^{3/2}\,|\theta-\theta_M|^{3/2}\right)^{2/3} \le M_\theta\,.$$

Thus, we can reproduce the bound (7.2) at least in a weaker norm.

Since no full justification of (7.2) is available, we are now interested in the question whether the existence results of the first section extend to the genuine Boussinesq ansatz (3.23).

Theorem 7.2.3. Let the assumptions of Theorem 7.1.1 be satisfied, but replace the growth assumption (7.1) on f by the condition (3.23). If the number α is sufficiently small with respect to the other data, the existence result of Theorem 7.1.1 remains valid.

7.3 A uniqueness result for strong solutions

As usual when the Navier-Stokes system is involved, there is a gap between the classes of functions in which existence and uniqueness can respectively be proved (see for example Lions [1969], Ch. 1, Section 6).

In this section, we prove the following simple result:

Lemma 7.3.1. Let the coefficients η, r, κ be piecewise constant, and assume that the force term $f \in [C(\mathbb{R})]^3$ is Lipschitz continuous. Assume that $\Sigma \in \mathcal{C}^{1,\alpha}$ for some $\alpha > 1/4$.

Then, the problem (P) has at most one weak solution $\{v,\, H,\, \theta\}$ such that $\theta \in W_2^{1,0}(Q)$ and

$$C^* := \operatorname*{ess\ sup}_{t\in[0,T]} \left\{\|v(t)\|_{[L^4(Q_1)]^3} + \|\operatorname{curl} H(t)\|_{[L^3(\tilde{Q})]^3} + \|\theta(t)\|_{L^4(\Sigma)}\right\} < \infty\,. \qquad (7.46)$$

Proof. We assume that $\{v_1,\, H_1,\, \theta_1\}$ and $\{v_2,\, H_2,\, \theta_2\}$ are two weak solutions with the additional property (7.46). We test in the Navier-Stokes equations with $v_1 - v_2$ and in Maxwell's equations with $H_1 - H_2$. Using standard inequalities and the property (7.46), we prove that

$$
\begin{aligned}
&\frac{\rho_1}{2}\, \|v_1(t) - v_2(t)\|^2_{[L^2(\Omega_1)]^3} + \int_{Q_{1,t}} \frac{\eta}{2}\, |\nabla(v_1 - v_2)|^2 \\
&\quad + \int_{\tilde{\Omega}} \mu\, |H_1(t) - H_2(t)|^2 + \int_{\tilde{Q}_t} \frac{r_l}{2}\, |\operatorname{curl}(H_1 - H_2)|^2 \\
&\qquad \le C\, (\|v_1 - v_2\|^2_{[L^2(Q_{1,t})]^3} + \|H_1 - H_2\|^2_{[L^2(\tilde{Q}_t)]^3} + \|f(\theta_1) - f(\theta_2)\|^2_{[L^2(Q_{1,t})]^3})\,. \qquad (7.47)
\end{aligned}
$$

This part of the proof being fairly standard, we do not execute it in greater detail. The difference $\theta_1 - \theta_2$ satisfies for almost all $t \in]0, T[$

$$
\begin{aligned}
&\int_0^t \langle \rho\, c_V (\theta_1 - \theta_2)', \xi \rangle + \int_{Q_t} \rho\, c_V\, v \cdot \nabla(\theta_1 - \theta_2)\, \xi + \int_{Q_t} \kappa\, \nabla(\theta_1 - \theta_2) \cdot \nabla \xi \\
&\qquad + \int_{\mathcal{S}_t} G(\sigma\, [\theta_1^4 - \theta_2^4])\, \xi = \int_{Q_t \setminus Q_{1,t}} r\, (|\operatorname{curl} H_1|^2 - |\operatorname{curl} H_2|^2)\, \xi\,.
\end{aligned}
$$

We test this relation with $\xi := \theta_1 - \theta_2$, we use the fact that G is selfadjoint, and the decomposition $G = \epsilon\, (I - \tilde{\mathbf{H}})$ of Lemma 4.1.14, (1). If follows that

$$
\begin{aligned}
&\int_\Omega \rho\, c_V\, |\theta_1(t) - \theta_2(t)|^2 + \int_{Q_t} \kappa\, |\nabla(\theta_1 - \theta_2)|^2 + \int_{\mathcal{S}_t} \sigma\, \epsilon\, [\theta_1^4 - \theta_2^4]\, (\theta_1 - \theta_2) \\
&\quad = \int_{\mathcal{S}_t} \sigma\, \epsilon\, [\theta_1^4 - \theta_2^4]\, \tilde{\mathbf{H}}(\theta_1 - \theta_2) + \int_{Q_t \setminus Q_{1,t}} r\, (|\operatorname{curl} H_1|^2 - |\operatorname{curl} H_2|^2)\, (\theta_1 - \theta_2)\,. \qquad (7.48)
\end{aligned}
$$

Now, we estimate the right-hand side of (7.48). First, in view of Lemma 4.1.14, (2), we have by assumption that the operator $\tilde{\mathbf{H}}$ is continuous from $L^4(\Sigma)$ into $C(\Sigma)$. We can thus write

$$
\begin{aligned}
&\left| \int_{\mathcal{S}_t} \sigma\, \epsilon\, [\theta_1^4 - \theta_2^4]\, \tilde{\mathbf{H}}(\theta_1 - \theta_2) \right| \\
&\quad \le 4\, \sigma \int_0^t \|\tilde{\mathbf{H}}(\theta_1 - \theta_2)(t)\|_{C(\Sigma)}\, \|\theta_1(t) - \theta_2(t)\|_{L^4(\Sigma)}\, (\|\theta_1(t)\|^3_{L^4(\Sigma)} + \|\theta_2(t)\|^3_{L^4(\Sigma)}) \\
&\qquad \le 4\, \sigma\, \|\tilde{\mathbf{H}}\|_{\mathcal{L}(L^4(\Sigma), C(\Sigma))} \int_0^t \|\theta_1(t) - \theta_2(t)\|^2_{L^4(\Sigma)}\, (\|\theta_1(t)\|^3_{L^4(\Sigma)} + \|\theta_2(t)\|^3_{L^4(\Sigma)}) \\
&\qquad\quad \le c\, {C^*}^3 \int_0^t \|\theta_1(t) - \theta_2(t)\|^2_{L^4(\Sigma)}\,,
\end{aligned}
$$

thanks also to (7.46). Now, we use the inequality

$$
\|\theta_1(t) - \theta_2(t)\|^2_{L^4(\Sigma)} \le \delta\, \|\nabla(\theta_1(t) - \theta_2(t))\|^2_{[L^2(\Omega)]^3} + c_\delta\, \|\theta_1(t) - \theta_2(t)\|^2_{L^2(\Omega)}\,,
$$

which is valid for all $\delta > 0$ with a constant c_δ depending on δ. For a suitable choice of δ, it follows that

$$\left|\int_{\mathcal{S}_t} \sigma\,\epsilon\,[\theta_1^4 - \theta_2^4]\,\tilde{\mathbf{H}}(\theta_1 - \theta_2)\right| \le \frac{\kappa_l}{4}\int_{Q_t} |\nabla(\theta_1 - \theta_2)|^2 + C\int_{Q_t} |\theta_1 - \theta_2|^2 . \tag{7.49}$$

On the other hand, we have

$$\begin{aligned}
&\left|\int_{Q_t\setminus Q_{1,t}} r\,(|\operatorname{curl} H_1|^2 - |\operatorname{curl} H_2|^2)\,(\theta_1 - \theta_2)\right| \\
&\le r_u \int_0^t \|\operatorname{curl}(H_1(t_1) + H_2(t_1))\|_{[L^3]^3}\,\|\operatorname{curl}(H_1(t_1) - H_2(t_1))\|_{[L^2]^3}\,\|\theta_1(t_1) - \theta_2(t_1)\|_{L^6} \\
&\quad \le 2\,r_u\,c_0\,C^* \int_0^t \|\operatorname{curl}(H_1(t_1) - H_2(t_1))\|_{[L^2(\Omega)]^3}\,\|\theta_1(t_1) - \theta_2(t_1)\|_{W^{1,2}(\Omega)} ,
\end{aligned}$$

with the constant c_0 of the continuous embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$. Applying again Young's inequality, we obtain that

$$\begin{aligned}
&\left|\int_{Q_t\setminus Q_{1,t}} r\,(|\operatorname{curl} H_1|^2 - |\operatorname{curl} H_2|^2)\,(\theta_1 - \theta_2)\right| \\
&\quad \le \frac{\kappa_l}{4}\int_{Q_t} |\nabla(\theta_1 - \theta_2)|^2 + C\left(\int_{Q_t} |\theta_1 - \theta_2|^2 + \int_{\tilde{Q}_t} |\operatorname{curl}(H_1 - H_2)|^2\right)
\end{aligned} \tag{7.50}$$

The relations (7.48), (7.49) and (7.50) imply that

$$\begin{aligned}
&\int_\Omega \rho\,c_V\,|\theta_1(t) - \theta_2(t)|^2 + \int_{Q_t} \frac{\kappa_l}{2}\,|\nabla(\theta_1 - \theta_2)|^2 + \int_{\mathcal{S}_t} \sigma\,\epsilon\,[\theta_1^4 - \theta_2^4]\,(\theta_1 - \theta_2) \\
&\quad \le C\left(\int_{Q_t} |\theta_1 - \theta_2|^2 + \int_{\tilde{Q}_t} |\operatorname{curl}(H_1 - H_2)|^2\right) .
\end{aligned}$$

Using also (7.47) and the fact that f is Lipschitz continuous, we now find that

$$\begin{aligned}
&\int_\Omega \rho\,c_V\,|\theta_1(t) - \theta_2(t)|^2 + \int_{Q_t} \frac{\kappa_l}{2}\,|\nabla(\theta_1 - \theta_2)|^2 + \int_{\mathcal{S}_t} \sigma\,\epsilon\,[\theta_1^4 - \theta_2^4]\,(\theta_1 - \theta_2) \\
&\quad \le \tilde{C}\left(\|\theta_1 - \theta_2\|^2_{L^2(Q_t)} + \|v_1 - v_2\|^2_{[L^2(Q_{1,t})]^3} + \|H_1 - H_2\|^2_{[L^2(\tilde{Q}_t)]^3}\right) .
\end{aligned}$$

Exploiting (7.47) again, we finally arrive for almost all $t \in]0, T[$ at

$$\begin{aligned}
&\|\theta_1(t) - \theta_2(t)\|^2_{L^2(\Omega)} + \|v_1(t) - v_2(t)\|^2_{[L^2(\Omega_1)]^3} + \|H_1(t) - H_2(t)\|^2_{[L^2(\tilde{\Omega})]^3} \\
&\quad \le \bar{C}\left(\|\theta_1 - \theta_2\|^2_{L^2(Q_t)} + \|v_1 - v_2\|^2_{[L^2(Q_{1,t})]^3} + \|H_1 - H_2\|^2_{[L^2(\tilde{Q}_t)]^3}\right) .
\end{aligned}$$

The Gronwall lemma proves the claim. □

7.4 Simplified models that lead to more regular solutions

The modification of the model that we want discuss is the approximation of low-magnetic Reynolds numbers described in the paragraph 2.3. This is well-suited for cases in which the caracteristic number $Rm := \mu\, s\, V\, d$ (see the appendix B) is sufficiently small for allowing to neglect the motion induced current $v \times B$ in Ohm's law (3.28). This case is relevant for many technical applications and for the numerical practice (see for example Lechner et al. [2007], Peterson [1988]).

We also make the assumption that the electrical conductivity $\mathfrak{s}$ does not significantly depend on temperature, and

$$\mathfrak{s}_i \in \mathcal{C}(\overline{\tilde{\Omega}_i}) \quad \text{for } i = 0, \ldots, m\,. \tag{7.51}$$

Obviously, Maxwell's equations decouple from the rest of the problem. To make things simpler, we can assume that the evolution of the electromagnetic fields is time-harmonic, which is possible if we drop the condition (3.38). In the present section, we therefore discuss the modification of the problem (P) that arises from replacing (3.28) by the relation $j = \mathfrak{s}\, E$, and from dropping the condition (3.38), and we denote by (P') this new problem.

Theorem 7.4.1. Let the assumptions of Theorem 7.1.1 be satisfied. Assume that the coefficients satisfy the usual assumptions of the section 3.1.3, supplemented by (7.51) in the whole of the electrical conductors, and let (3.50) be satisfied in connection with (5.17). Assume in addition that the amplitude of the given current satisfies $\operatorname{curl} j_0 \in [L^2(\tilde{\Omega}_{c_0})]^3$.

Then, even if the force term f has arbitrary polynomial growth at infinity, there exists a weak solution of (P'), such that $\operatorname{curl} H \in [L^{\infty,6}(Q_c)]^3$, and such that $\theta \in L^\infty(Q)$.

Remark 7.4.2. Theorem 7.4.1 does not make any requirement about the growth of the force term f. We could as well take the ohmic dissipation $|\operatorname{curl} H|^2$ into account in the fluid.

The remainder of the section is devoted to the proof of Theorem 7.4.1. First, we have a regularity result.

Lemma 7.4.3. Let the assumptions of Theorem 7.4.1 be satisfied. Let $H \in \mathcal{H}_\mu(\tilde{Q})$ be such that for all $\psi \in \mathcal{H}_\mu(\tilde{Q})$,

$$-\int_{\tilde{Q}} \mu\, H \cdot \frac{\partial \psi}{\partial t} + \int_{\tilde{Q}} r\,(\operatorname{curl} H - j_g) \cdot \operatorname{curl} \psi = 0\,. \tag{7.52}$$

Then $\operatorname{curl} H$ belongs to $[L^{\infty,6}(Q_c)]^3$.

Proof. Since the current source j_g is harmonic in time, it is well-known that the decoupled Maxwell's equations can be solved in time-harmonic regime (see Bossavit [2004]). We easily show that the solution field H has the form $H(t,x) = \operatorname{Im}\Big(\exp(i\,\omega\, t)\, \tilde{H}(x)\Big)$, with $\tilde{H} \in \mathcal{H}_\mu(\tilde{\Omega})$. Therefore, we can write that

$$\int_{\tilde{Q}} r \ \operatorname{curl} H \cdot \operatorname{curl} \psi = -\int_{\tilde{Q}} \mu \frac{\partial H}{\partial t} \cdot \psi + \int_{\tilde{Q}} r\, j_g \cdot \operatorname{curl} \psi\,,$$

and we easily verify by means of Lemma A.4.3, that this relation holds true for all $\psi \in \mathcal{H}(\tilde{Q})$. For almost all $t \in]0,T[$ and $\tilde{\Omega}_i \subset \tilde{\Omega}_c$ arbitrary, and for all $\psi \in C_c^\infty(\tilde{\Omega}_i)$, we get

$$\int_{\tilde{\Omega}_i} r_i \operatorname{curl} H(t) \cdot \operatorname{curl} \psi = -\int_{\tilde{\Omega}_i} \mu \frac{\partial H}{\partial t}(t) \cdot \psi + \int_{\tilde{\Omega}_i} \operatorname{curl}(r\, j_g)\, \psi\,.$$

We introduce the abbreviation $w_{i,t} := r_i \operatorname{curl} H(t)$. We see that the last relation means nothing else than

$$\operatorname{curl} w_{i,t} = \mu \frac{\partial H}{\partial t}(t) + \operatorname{curl}(r\, j_g)(t) \in [L^2(\tilde{\Omega}_i)]^3\,. \tag{7.53}$$

On the other hand, we easily find that

$$\operatorname{div} w_{i,t} = \nabla r_i \cdot \operatorname{curl} H(t) \ \in L^2(\tilde{\Omega}_i)\,. \tag{7.54}$$

By Lemma A.4.1, we also know that

$$w_{i,t} \cdot \vec{n} = 0 \text{ on } \partial\tilde{\Omega}_i\,. \tag{7.55}$$

In the book Duvaut and Lions [1976], Ch. 7, Th. 6.1, it is proved that for a bounded domain Ω with $\partial\Omega \in \mathcal{C}^2$, and all $\psi \in [L^2(\Omega)]^3$ such that

$$\operatorname{curl} \psi \in [L^2(\Omega)]^3, \quad \operatorname{div} \psi \in L^2(\Omega), \quad \psi \cdot \vec{n} = 0 \text{ on } \partial\Omega\,,$$

one has $\psi \in [W^{1,2}(\Omega)]^3$ and the inequality

$$\|\psi\|_{[W^{1,2}(\Omega)]^3} \le c\,(\|\operatorname{curl}\psi\|_{[L^2(\Omega)]^3} + \|\operatorname{div}\psi\|_{[L^2(\Omega)]^3})\,,$$

with a constant c depending on Ω. Therefore, the results (7.53), (7.54) and (7.55) show that $w_{i,t} \in [W^{1,2}(\tilde{\Omega}_i)]^3$, and that

$$\|w_{i,t}\|_{[W^{1,2}(\tilde{\Omega}_i)]^3} \le c \left(\left\| \frac{\partial H}{\partial t}(t) \right\|_{[L^2(\tilde{\Omega}_i)]^3} + \|\operatorname{curl}(r\, j_g)\|_{[L^2(\tilde{\Omega}_i)]^3} + \|\nabla r_i \cdot \operatorname{curl} H(t)\|_{L^2(\tilde{\Omega}_i)} \right)\,.$$

By Sobolev's embedding results, we now find

$$\begin{aligned} r_l\, \|\operatorname{curl} H(t)\|_{[L^6(\tilde{\Omega}_i)]^3} &\le \|w_{i,t}\|_{[L^6(\tilde{\Omega}_i)]^3} \\ &\le c \left(\left\| \frac{\partial H}{\partial t}(t) \right\|_{[L^2(\tilde{\Omega}_i)]^3} + \|\operatorname{curl}(r\, j_g)\|_{[L^2(\tilde{\Omega}_i)]^3} + \|\nabla r_i \cdot \operatorname{curl} H(t)\|_{[L^2(\tilde{\Omega}_i)]^3} \right)\,. \end{aligned}$$

Since the right-hand side of the last expression is bounded in $L^\infty(0,T)$, the claim follows. □

Proof of Theorem 7.4.1. The Maxwell system is decoupled from the rest of the problem, and can be solved in time harmonic regime. For solving the system that consists of Navier-Stokes equations and the heat equation, we first construct approximate solutions using techniques similar to Propostion 7.1.2. In view of Lemma 7.4.3, we even do not need to apply a cutoff operator to the right-hand side of the heat equation. However, since we consider that f has arbitrary growth, we apply the cutoff operator $[\cdot]_{(\delta)}$ to $f(\theta)$.

In Druet [2008a], we proved that the solution operator of the time-dependent heat equation with non local radiation boundary condition and fixed right-hand side $\bar{f} \in L^r(Q)$ maps into $L^\infty(Q)$ if $r > 5/2$.

By Lemma 7.4.3, the right-hand side $|\operatorname{curl} H|^2$ clearly satisfies this last assumption. With the techniques of Druet [2008a], we then obtain a uniform bound of the approximating sequence $\{\theta_\delta\}$ in the norm of $V_0^{2,5}(Q)$ and of $L^\infty(Q)$. Passage to the limit occurs as in Proposition 7.1.4. $\square$

A similar result is valid for the *quasi-stationary approach* described in the paragraph 2.3, based on the argument of section 6.4.

Appendix A

Some tools and technical lemma

A.1 Identities of vector analysis

For vectors a, b, the Euclidean scalar product is defined as

$$a \cdot b := \sum_{i=1}^{3} a_i \, b_i \, . \tag{A.1}$$

The vector product ("cross" product) is defined by

$$a \times b := (a_2 \, b_3 - a_3 \, b_2, \, a_3 \, b_1 - a_1 \, b_3, \, a_1 \, b_2 - a_2 \, b_1)^T \, . \tag{A.2}$$

Commutations rules are given by

$$a \cdot (b \times c) = b \cdot (c \times a) = c \cdot (a \times b) \, . \tag{A.3}$$

For a scalar valued function $u : \mathbb{R}^3 \to \mathbb{R}$ and a vector field $v : \mathbb{R}^3 \to \mathbb{R}^3$, one introduces the differential operators

$$\operatorname{grad} u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3} \right)^T \, , \tag{A.4}$$

$$\operatorname{div} v = \sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i} \, , \tag{A.5}$$

$$\operatorname{curl} v = \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right)^T \, . \tag{A.6}$$

The $\nabla-$operator (to pronounce "nabla") is formally the vector $(\partial_{x_1}, \, \partial_{x_2}, \, \partial_{x_3})^T$. Therefore, one can write

$$\nabla u = \operatorname{grad} u, \quad \nabla \cdot v = \operatorname{div} v, \quad \nabla \times v = \operatorname{curl} v \, .$$

It is well-known that

$$\operatorname{curl} \operatorname{grad} u = 0 \, , \quad \operatorname{div} \operatorname{curl} v = 0 \, . \tag{A.7}$$

Often used in the context of Maxwell's equations is the formula

$$\operatorname{curl}\operatorname{curl} v = \operatorname{grad}\operatorname{div} v - \triangle v\,, \tag{A.8}$$

where $\triangle$ is the Laplace operator applied componentwise. For two vector fields v, w : $\mathbb{R}^3 \to \mathbb{R}^3$, we need the identity

$$\operatorname{grad}(v\cdot w) = (v\cdot\nabla)w + (w\cdot\nabla)v + v\times\operatorname{curl} w + w\times\operatorname{curl} v\,, \tag{A.9}$$

which also implies that

$$\operatorname{grad}(\frac{|v|^2}{2}) = (v\cdot\nabla)v + v\times\operatorname{curl} v\,. \tag{A.10}$$

We also have

$$\operatorname{curl}(v\times w) = v\ \operatorname{div} w - w\ \operatorname{div} v + (w\cdot\nabla)v - (v\cdot\nabla)w\,, \tag{A.11}$$

which implies for solenoidal vector fields v, w that

$$\operatorname{curl}(v\times w) = (w\cdot\nabla)v - (v\cdot\nabla)w\,. \tag{A.12}$$

Finally, we want to cite Gausses theorem for the divergence operator. If $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain, and if v, $w \in [\mathcal{C}^1(\overline{\Omega})]^3$ and $u \in \mathcal{C}^1(\overline{\Omega})$, then

$$\int_\Omega \operatorname{div} v\, u = \int_{\partial\Omega} v\cdot\vec{n}\, u\, d\,S - \int_\Omega v\cdot\operatorname{grad} u\,, \tag{A.13}$$

where $\vec{n}$ is the normal vector pointing outwards to Ω, and S is the surface measure. For the rotation operator, Gausses theorem takes the form

$$\int_\Omega \operatorname{curl} v\cdot w\, dx = \int_{\partial\Omega} (v\times w)\cdot\vec{n}\, d\,S + \int_\Omega v\times\operatorname{curl} w\,. \tag{A.14}$$

A.2 Energy estimates in a nonlinear PDE with right-hand side in L^1

The existence of weak solutions in the class $\bigcap_{1\leq p<\frac{n}{n-1}} W^{1,p}(\Omega)$ for the Laplace equation with L^1-right-hand side in a bounded domain $\Omega \subset \mathbb{R}^n$ was first proved in Stampacchia [1965].

In Boccardo and Gallouët [1992a] and Rakotoson [1991], techniques for obtaining estimates involving the L^1-norm of the right-hand side were developed. A typical application of these techniques is the study of systems with energy balance with source terms having quadratic growth in the first derivative of the unknowns (see Naumann [2005], Druet [2007b], and many other examples).

A.2.1 Elliptic problems

Lemma A.2.1. For a $p < 2$, let $\theta \in W^{1,p}_\Gamma(\Omega)$. Suppose that there exists a constant $C_1 > 0$ such that for all $\delta \in]\,0\,,1\,[$ one has

$$\int_\Omega \frac{|\nabla\theta|^2}{(1+\theta)^{1+\delta}} \le \frac{C_1}{\delta}\,.$$

Then one has

$$\int_\Omega |\nabla\theta|^p \le c_{1,p}\,\mathrm{meas}(\Omega)^{\frac{2-p}{2}}\; C_1^{\frac{p}{2}} + c_{2,p}\,c_0^{6-3p}\,C_1^{3-p}\,, \tag{A.15}$$

where c_0 is the embedding constant of $W^{1,p}_\Gamma(\Omega) \hookrightarrow L^{p^*}(\Omega)$ (p^* = Sobolev embedding exponent), and the constants c_1, c_2 depend only on p.

Proof. We can write

$$\int_\Omega |\nabla\theta|^p = \int_\Omega \frac{|\nabla\theta|^p}{(1+\theta)^{(1+\delta)\,\frac{p}{2}}}\,(1+\theta)^{(1+\delta)\,\frac{p}{2}} \le \left(\frac{C_1}{\delta}\right)^{\frac{p}{2}} \left(\int_\Omega |1+\theta|^{(1+\delta)\,\frac{p}{2}\,\frac{2}{2-p}}\right)^{\frac{2-p}{2}}\,.$$

If we denote by p^* the Sobolev embedding exponent, we find that for the choice $\delta = \frac{3-2p}{3-p}$, we have

$$\left(\int_\Omega |1+\theta|^{(1+\delta)\,\frac{p}{2}\,\frac{2}{2-p}}\right)^{\frac{2-p}{2}} = \|1+\theta\|_{L^{p^*}(\Omega)}^{\frac{(2-p)\,p^*}{2}} \le \mathrm{meas}(\Omega)^{\frac{2-p}{2}} + c_0^{\frac{(2-p)\,p^*}{2}}\,\|\nabla\theta\|_{L^p(\Omega)}^{\frac{(2-p)\,p^*}{2}}\,.$$

Defining $r := \frac{6-2p}{6-3p} > 1$, and applying Young's inequality, we obtain that

$$\int_\Omega |\nabla\theta|^p \le \left(\frac{C_1}{\delta}\right)^{\frac{p}{2}} \mathrm{meas}(\Omega)^{\frac{2-p}{2}} + \frac{p\,c_0^{6-3p}}{6-2p}\left(\frac{3\,(2-p)}{3-p}\right)^{\frac{6-3p}{p}}\left(\frac{C_1}{\delta}\right)^{3-p} + \frac{\|\nabla\theta\|^p_{L^p(\Omega)}}{2}\,.$$

□

Lemma A.2.2. Let $(X,\,\mathcal{A},\,\mu)$ be a measurable space such that $\mu(X) < \infty$. For a measurable function $u\,:\,X \to \mathbb{R}$ and $1 < p < \infty$, define

$$[u]_{L^p_w(X)} := \sup_{t>0}\left\{t\,\mu\big(\{x \in X\,:\,|u(x)| > t\}\big)^{\frac{1}{p}}\right\}.$$

Then for all $1 < p < \infty$ and all $0 < \epsilon < p-1$, one has the inequality

$$\|u\|_{L^{p-\epsilon}(X,\mathcal{A},\mu)} \le \left(\frac{p}{\epsilon}\right)^{\frac{1}{p-\epsilon}} \Big(\mu(X)\Big)^{\frac{\epsilon}{p\,(p-\epsilon)}}\,[u]_{L^p_w(X)}\,.$$

Proof. See Kufner et al. [1977], paragraph 2. 18. □

A.2.2 Parabolic problems

Lemma A.2.3. Let $q \geq 1$ be a real number. Let $1 \leq p \leq q^*$ (q^* = Sobolev embedding exponent). Let $\xi \in W^{1,0}_p(Q) \cap L^{\infty,1}(Q)$.

1 If $q < 3$, and if we define $r := \frac{(4q-3)\,p}{3\,(p-1)}$, then we find a positive constant $C = C(p,q)$ such that

$$\|\xi\|_{L^{r,p}(Q)} \leq C\,\|\xi\|^{1-\alpha}_{L^{\infty,1}(Q)}\,\|\nabla\xi\|^{\alpha}_{L^q(Q)}\,,$$

with $\alpha = \frac{3\,q\,(p-1)}{p\,(4q-3)}$.

2 If $q > 3$, and if we define $r := \frac{q\,p}{p-1}$, then we find a positive constant $C = C(p,q)$ such that

$$\|\xi\|_{L^{r,p}(Q)} \leq C\,\|\xi\|^{1/p}_{L^{\infty,1}(Q)}\,\|\nabla\xi\|^{1/p'}_{L^q(Q)}\,.$$

Proof. For $0 \leq \lambda \leq 1$ we can write

$$\|\xi(t)\|^p_{L^p(\Omega)} \leq \|\xi(t)\|^{\lambda}_{L^1(\Omega)}\left(\int_\Omega |\xi(t)|^{\frac{p-\lambda}{1-\lambda}}\right)^{1-\lambda}\,.$$

If we fix λ such that

$$\frac{p-\lambda}{1-\lambda} = q^*\,, \tag{A.16}$$

we will find be able to write

$$\|\xi(t)\|^p_{L^p(\Omega)} \leq c\,\|\xi(t)\|^{\lambda}_{L^1(\Omega)}\,\|\nabla\xi(t)\|^{p-\lambda}_{L^q(\Omega)}\,.$$

Thus

$$\|\xi\|^r_{L^{r,p}(Q)} \leq c\,\|\xi\|^{\lambda\,r/p}_{L^{\infty,1}(Q)}\int_0^T \|\nabla\xi\|^{\frac{(p-\lambda)\,r}{p}}_{L^q(\Omega)}\,.$$

Under the condition

$$\frac{(p-\lambda)\,r}{p} = q\,, \tag{A.17}$$

we will prove the estimate

$$\|\xi\|_{L^{r,p}(Q)} \leq C\,\|\xi\|^{\lambda/p}_{L^{\infty,1}(Q)}\,\|\nabla\xi\|^{q/r}_{L^q(Q)}\,.$$

The relations (A.16) and (A.17) determine λ, r uniquely. The assertion follows from this elementary calculation. $\square$

Proposition A.2.4. For $n \in \mathbb{N}$ and $u \in W_p^{1,0}(Q) \cap L^{\infty,1}(Q)$, define

$$B_n := \left\{ (t,x) \in [0,T] \times \Omega \,\middle|\, n \le |u(t,x)| < n+1 \right\}.$$

Suppose that there exists a positive constant C_* such that

$$\sup_{n\in\mathbb{N}} \int_{B_n} |\nabla u|^p \, dx \, dt \le C_* \,.$$

(1) If $p < \frac{15}{4}$, then for all $1 \le q < p - \frac{3}{4}$, we can find positive constants c_1, c_2 that depends only on Ω, q, p, such that

$$\| \nabla u \|_{L^q(Q)} \le c_1 + c_2 \, \| u \|^s_{L^{\infty,1}(Q)} \, C_*^{1/q} \,,$$

with $s = \frac{p-q}{3\,q}$.

(2) If $p > \frac{15}{4}$, then for all $3 < q < \frac{p-2+\sqrt{(p-2)^2+4\,p}}{2}$, we find positive constants c_1, c_2 that depends only on Ω, q, p such that

$$\| \nabla u \|_{L^q(Q)} \le c_1 + c_2 \, \| u \|^s_{L^{\infty,1}(Q)} \, C_*^{1/q} \,,$$

with $s = \frac{p-q}{q^2}$.

Proof. Similar results were proved for the first time in Boccardo and Gallouët [1992a]. We follow the argumentation of Lewandowski [1997], making only slight modifications. For $n \ge 1$ and $q < p$ one has the estimates

$$\int_{B_n} |\nabla u|^q \le \text{meas}(B_n)^\mu \left(\int_{B_n} |\nabla u|^p \right)^{q/p} , \qquad \text{meas}(B_n) \le \frac{1}{n^r} \int_{B_n} |u|^r \,,$$

where $r \ge 1$ is a number we want to fix later, and $\mu := \frac{p-q}{p}$. By assumption, we now get for all $n \ge 1$

$$\int_{B_n} |\nabla u|^q \le \frac{C_*^{q/p}}{n^{r\,\mu}} \left(\int_{B_n} |u|^r \right)^\mu .$$

We estimate $\text{meas}(B_0) \le \text{meas}(\Omega)$. Summing up for $n \in \mathbb{N}$ yields

$$\begin{aligned} \int_Q |\nabla u|^q &\le \text{meas}(\Omega)^\mu \, C_*^{q/p} + C_*^{q/p} \left(\sum_{n=1}^\infty \frac{1}{n^{r\mu}} \left(\int_{B_n} |u|^r \right)^\mu \right) \\ &\le \text{meas}(\Omega)^\mu \, C_*^{q/p} + C_*^{q/p} \left(\sum_{n=1}^\infty \frac{1}{n^{\frac{r\mu}{1-\mu}}} \right)^{1-\mu} \left(\int_Q |u|^r \right)^\mu , \end{aligned}$$

where we justify the last inequality by applying Hölder's inequality. Under the condition

$$\frac{r\mu}{1-\mu} > 1 \,, \tag{A.18}$$

the series $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{r\mu}{1-\mu}}} =: A(p,q,r)$ converges, and we can write the inequality

$$\int_Q |\nabla u|^q \leq \mathrm{meas}(\Omega)^\mu \, C_*^{q/p} + C_*^{q/p} \, A^{1-\mu} \left(\int_Q |u|^r \right)^\mu . \tag{A.19}$$

Now, define

$$q^* := \begin{cases} \frac{3q}{3-q} & \text{if } q < 3\,, \\ \infty & \text{if } q > 3\,. \end{cases}$$

We set $\lambda := \frac{q^*-r}{q^*-1}$. Observe that $\lambda \leq 1$, as long as $r > 1$. This allows us to write

$$\int_\Omega |u|^\lambda \, |u|^{r-\lambda} \;\; \leq \;\; \left(\int_\Omega |u| \right)^\lambda \left(\int_\Omega |u|^{r-\lambda/1-\lambda} \right)^{1-\lambda} \leq \| \, u \, \|^\lambda_{L^{\infty,1}(Q)} \left(\int_\Omega |u|^{r-\lambda/1-\lambda} \right)^{1-\lambda} .$$

Substituting again for λ and integrating, we obtain

$$\int_0^T \int_\Omega |u|^r \leq \| \, u \, \|^{\frac{q^*-r}{q^*-1}}_{L^{\infty,1}(Q)} \int_0^T \left(\int_\Omega |u|^{q^*} \right)^{\frac{r-1}{q^*-1}} .$$

If we insert this last result in (A.19), we get

$$\int_Q |\nabla u|^q \leq \mathrm{meas}(\Omega)^\mu \, C_*^{q/p} + C_*^{q/p} \, A^{1-\mu} \left\{ \| \, u \, \|^{\frac{q^*-r}{q^*-1}}_{L^{\infty,1}(Q)} \left[\int_\Omega \left(\int_\Omega |u|^{q^*} \right)^{\frac{r-1}{q^*-1}} \right] \right\}^\mu .$$

Now, using Sobolev's theorems we obtain

$$\int_Q |\nabla u|^q \leq \mathrm{meas}(\Omega)^\mu \, C_*^{q/p} + \tilde{c} \, C_*^{q/p} \, A^{1-\mu} \;\; \| \, u \, \|^{\frac{(q^*-r)\,\mu}{q^*-1}}_{L^{\infty,1}(Q)} \left[\int_0^T \left(\int_\Omega |\nabla u|^q \right)^{\frac{q^*\,(r-1)}{q\,(q^*-1)}} \right]^\mu , \tag{A.20}$$

with the embedding constant $\tilde{c}$. We search for $r > 1$ such that (A.18) is fullfilled and that

$$\frac{q^*\,(r-1)}{q\,(q^*-1)} = 1\,. \tag{A.21}$$

In view of (A.21), we see that r must satisfy

$$r = \begin{cases} \frac{4}{3}\,q & \text{if } q < 3\,, \\ q+1 & \text{if } q > 3\,. \end{cases}$$

Thus, after elementary calculations, we see that (A.18) will be satisfied by $q < 3$ if

$$q < p - \frac{3}{4}\,,$$

and by $q > 3$ as long as

$$q < \frac{p - 2 + \sqrt{(p-2)^2 + 4p}}{2}\,.$$

Note now that $\frac{p-2+\sqrt{(p-2)^2+4p}}{2} > 3 \Longleftrightarrow p > \frac{15}{4}$. Thus, we will be able to choose q in the range

$$q \in \begin{cases} [1, p-3/4[& \text{if } p < \frac{15}{4}\,, \\ \left]3, \frac{p-2+\sqrt{(p-2)^2+4p}}{2}\right[& \text{if } p > \frac{15}{4}\,. \end{cases}$$

Using Young's inequality, we get, for this choices, from (A.20)

$$\int_Q |\nabla u|^q \leq \mathrm{meas}(\Omega)^\mu\, C_*^{q/p} + c_\gamma \left[c\, C_*^{q/p}\, A^{1-\mu} \ \|\, u\, \|_{L^{\infty,1}(Q)}^{\frac{(q^*-r)\,\mu}{q^*-1}} \right]^{\frac{1}{1-\mu}} + \gamma \int_Q |\nabla u|^q\,,$$

where γ is an arbitrary small positive number. It follows that

$$\int_Q |\nabla u|^q \leq c_1 + c_2\, C_*^{\frac{q}{p\,(1-\mu)}} \ \|\, u\, \|_{L^{\infty,1}(Q)}^{\frac{(q^*-r)\,\mu}{(q^*-1)\,(1-\mu)}}\ . \tag{A.22}$$

We easily verify that

$$\frac{(q^*-r)\,\mu}{(q^*-1)\,(1-\mu)} = \begin{cases} \frac{p-q}{3} & \text{if } p < \frac{15}{4}\,, \\ \frac{p-q}{q} & \text{if } p > \frac{15}{4}\,, \end{cases}$$

as well as $\frac{q}{p\,(1-\mu)} = 1$. This proves the claim. □

The following Lemma gives the same estimates as Proposition A.2.4, but with another technique.

Lemma A.2.5. Let $1 < p < \infty$ be fixed. We suppose that θ, $\theta_g \in W_p^{1,0}(Q) \cap L^{\infty,1}(Q)$ are such that $\theta - \theta_g = 0$ on $]0,T[\times\partial\Omega$. Suppose that there exists a constant $C_1 > 0$, such that for all $\gamma \in]0,1[$

$$\int_Q \frac{|\nabla\theta|^s}{(1+|\theta-\theta_g|)^{1+\gamma}} \leq \frac{C_1}{\gamma}\,.$$

Then, we find the following estimates:

1 If $s = 2$, then for all $q \in \left[1, \frac{4\,s-3}{4}\right[$, we find a positive constant $c = c(q)$ such that

$$\|\nabla\theta\|_{L^q(Q)} \leq c \left(\mathrm{meas}(Q)^{\frac{2-q}{2\,q}}\, C_1^{1/2} + \|\theta-\theta_g\|_{L^{\infty,1}(\Omega)}^{\frac{2-q}{3\,q}}\, C_1^{1/q} + \|\nabla\theta_g\|_{L^q(Q)} \right)\,.$$

2 If $s > \frac{15}{4}$, then for all $q \in \left]\frac{s-3+\sqrt{(s-3)^2+4s}}{2}\,,\ \frac{s-2+\sqrt{(s-2)^2+4s}}{2}\right[$, we find a positive constant $c = c(q,s)$ such that

$$\|\nabla\theta\|_{L^q(Q)} \leq c \left(\mathrm{meas}(Q)^{\frac{s-q}{s\,q}}\, C_1^{1/s} + \|\theta-\theta_g\|_{L^{\infty,1}(\Omega)}^{\frac{s-q}{q^2}}\, C_1^{1/q} + \|\nabla\theta_g\|_{L^q(Q)} \right)\,.$$

Proof. We show the estimate (1). The point (2) follows by the same arguments. We have from Hölder's inequality that

$$\int_Q |\nabla\theta|^q = \int_Q \frac{|\nabla\theta|^q}{(1+|\theta-\theta_g|)^{(1+\gamma)\,\frac{q}{2}}}\,(1+|\theta-\theta_g|)^{(1+\gamma)\,\frac{q}{2}}$$
$$\leq \left(\frac{C_1}{\gamma}\right)^{q/2}\,\|1+|\theta-\theta_g|\|^{\frac{(1+\gamma)\,q}{2}}_{L^{\frac{(1+\gamma)\,q}{2-q}}(Q)}\,.$$

It follows that

$$\int_Q |\nabla\theta|^q \leq \left(\frac{C_1}{\gamma}\right)^{q/2}\,\left(\text{meas}(Q)^{\frac{2-q}{2}} + \|\theta-\theta_g\|^{\frac{(1+\gamma)\,q}{2}}_{L^{\frac{(1+\gamma)\,q}{2-q}}(Q)}\right).$$

Now, we define $p := \frac{4\,q}{3}$. By Lemma A.2.3, we find the estimate

$$\|\theta-\theta_g\|_{L^p(Q)} \leq C\,\|\theta-\theta_g\|^{1/4}_{L^{\infty,1}(Q)}\,\|\nabla(\theta-\theta_g)\|^{3/4}_{L^q(Q)}\,.$$

We choose

$$\gamma := \frac{5-4\,q}{3} \Longrightarrow \frac{(1+\gamma)\,q}{2-q} = \frac{4\,q}{3}\,.$$

We verify easily that $\gamma\in]0,1[$, whenever q is in the range fixed by the assertion. Therefore, we have

$$\|\theta-\theta_g\|^{\frac{(1+\gamma)\,q}{2}}_{L^{\frac{(1+\gamma)\,q}{2-q}}(Q)} \leq \bar{c}\left\{\|\theta-\theta_g\|^{1/4}_{L^{\infty,1}(Q)}\,\|\nabla(\theta-\theta_g)\|^{3/4}_{L^q(Q)}\right\}^{\frac{(1+\gamma)\,q}{2}}\,.$$

It follows that

$$\int_Q |\nabla\theta|^q \leq \left(\frac{C_1}{\gamma}\right)^{q/2}\,\left(\text{meas}(Q)^{\frac{2-q}{2}} + \bar{c}\,\|\theta-\theta_g\|^{\frac{(1+\gamma)\,q}{8}}_{L^{\infty,1}(Q)}\,\|\nabla(\theta-\theta_g)\|^{\frac{3\,(1+\gamma)\,q}{8}}_{L^q(Q)}\right).$$

Using Young's inequality, we will obtain

$$\int_Q |\nabla\theta|^q \leq \left(\frac{C_1}{\gamma}\right)^{q/2}\,\left(\text{meas}(Q)^{\frac{2-q}{2}} + c\,\|\theta-\theta_g\|^{\frac{2-q}{3}}_{L^{\infty,1}(Q)}\,\frac{C_1}{\gamma} + \frac{1}{2}\,\|\nabla(\theta-\theta_g)\|^q_{L^q(Q)}\right).$$

The claim follows. □

A.3 Steklov averaging

We recall a recurrent tool that we use for the analysis of parabolic problems. For a function u defined on the cylinder $Q :=]0,\,T[\times\Omega$, we can introduce for all $h\in]0,T[$

$$u_{(h)}(x,t) := \frac{1}{h}\int_t^{t+h} u(x,\tau)\,d\tau\,.$$

The function $u_{(h)}$ is called the *Steklov averaging* of u. It belongs to $W_2^1(Q_{T-h})$, whenever u belongs to $W_2^{1,0}(Q)$. The fundamental properties of Steklov averaging are listed in Ladyzenskaja et al. [1968], Chapter II, paragraph 4. The notation

$$u_{(\underline{h})}(x,t) \quad := \quad \frac{1}{h}\int_{t-h}^{t} u(x,\tau)\,d\tau\,,$$

makes sense as soon as we extend u, for instance by zero, to interval $[-h,0]$. For functions $u,\eta : Q \to \mathbb{R}$, such that η vanishes in the intervals $[-h,0]$ and $[T-h,T]$, and such that $\int_Q u\,\eta\,dx\,dt < \infty$, the formula

$$\int_Q u\,\eta_{(\underline{h})}\,dx\,dt \quad = \quad \int_Q u_{(h)}\,\eta\,dx\,dt\,, \tag{A.23}$$

is valid.

Lemma A.3.1. Let ξ_1, $\xi_2 \in L^1(Q)$, and assume that for some $p > 1$, $\xi_3 \in [L^p(Q)]^3$. Denote as usual by p' the conjugated exponent to p, and let $u \in W^{1,0}_{p',\mathcal{C}}(Q) \cap C(0,T;L^1(\Omega))$ satisfy

$$-\int_Q u\,\frac{\partial\psi}{\partial t} = \int_Q \xi_1\,\psi + \xi_3\cdot\nabla\psi + \int_{\mathcal{S}} \xi_2\,\psi \tag{A.24}$$

for all $\psi \in C_c^\infty(0,T;C^\infty(\Omega))$ that vanishes on $\mathcal{C}$.

Let $g : \mathbb{R} \to \mathbb{R}$ be a globally Lipschitz continuous and bounded function that satisfies $g(0) = 0$, and let F denote the primitive function of g that vanishes in zero. Then for all $t_1 < T$, the identity

$$\int_\Omega F(u(t_1)) = \int_\Omega F(u(0)) + \int_{Q_{t_1}} \xi_1\,g(u) + \xi_3\cdot\nabla g(u) + \int_{\mathcal{S}_{t_1}} \xi_2\,g(u)$$

is valid

Proof. We denote by $C_\Gamma^\infty(\Omega)$ the set of all smooth functions in Ω that vanish on Γ.

We consider $t_1 < T$ arbitrary, and choose some positive number $h < T - t_1$.

For an arbitrary $\tilde\psi \in C_c^\infty(0,t_1;C_\Gamma^\infty(\Omega))$ that we extend by zero into $[t_1,T]$ and $[-h,0]$, the test function $\psi := \tilde\psi_{(\underline{h})}$ can be used in (A.24).

Transferring for each integral the Steklov averaging according to (A.23), we obtain that

$$\int_Q \frac{\partial u_{(h)}}{\partial t}\,\tilde\psi = \int_Q (\xi_1)_{(h)}\,\tilde\psi + (\xi_3)_{(h)}\cdot\nabla\tilde\psi + \int_{\mathcal{S}} (\xi_2)_{(h)}\,\tilde\psi\,,$$

for all $\tilde\psi \in C_c^\infty(0,t_1;C_\Gamma^\infty(\Omega))$.

Now, we approximate the function $g(u_{(h)})$ in the norm of $W^{1,0}_{p',\mathcal{C}}(Q_{t_1})$ by a sequence $\{\tilde\psi_k\} \subset C_c^\infty(0,t_1;C_\Gamma^\infty(\Omega))$. Passing to the limit $k\to\infty$, we obtain that

$$\int_{Q_{t_1}} \frac{\partial u_{(h)}}{\partial t}\,g(u_{(h)}) = \int_{Q_{t_1}} (\xi_1)_{(h)}\,g(u_{(h)}) + (\xi_3)_{(h)}\cdot\nabla g(u_{(h)}) + \int_{\mathcal{S}_{t_1}} (\xi_2)_{(h)}\,g(u_{(h)})\,.$$

Now, we observe that

$$\frac{\partial u_{(h)}}{\partial t}\, g(u_{(h)}) = \frac{\partial}{\partial t}\, F\big(u_{(h)}\big)\,,$$

so that the last relation is equivalent to the equation

$$\int_\Omega F(u_{(h)}(t_1)) = \int_\Omega F(u_{(h)}(0)) \\ + \int_{Q_{t_1}} (\xi_1)_{(h)}\, g(u_{(h)}) + (\xi_3)_{(h)} \cdot \nabla g(u_{(h)}) + \int_{\mathcal{S}_{t_1}} (\xi_2)_{(h)}\, g(u_{(h)})\,.$$

Since $u \in C([0,T]; L^1(\Omega))$, we have for all $t \in [0,T]$ and $h \to 0$ that

$$u_{(h)}(t) \longrightarrow u(t) \quad \text{ in } L^1(\Omega)\,.$$

As the function g is globally bounded, its primitive F has at most linear growth, which implies that

$$F(u_{(h)}(t)) \longrightarrow F(u(t)) \quad \text{ in } L^1(\Omega)\,,$$

for all $t \in [0,T]$.

Now, we check the convergence of the other integral terms. We know that

$$u_{(h)} \longrightarrow u \quad \text{ in } L^1(Q_{t_1}) \text{ and in } L^1(\mathcal{S}_{t_1})\,.$$

Therefore, we can extract a subsequence such that

$$u_{(h)} \longrightarrow u \text{ almost everywhere in } Q_{t_1} \text{ and on } \mathcal{S}_{t_1}\,.$$

Since

$$(\xi_1)_{(h)} \longrightarrow \xi_1 \quad \text{ in } L^1(Q_{t_1}), \qquad (\xi_2)_{(h)} \longrightarrow \xi_2 \quad \text{ in } L^1(\mathcal{S}_{t_1})\,,$$

we easily verify that, as $h \to 0$,

$$\int_{Q_{t_1}} (\xi_1)_{(h)}\, g(u_{(h)}) \longrightarrow \int_{Q_{t_1}} \xi_1\, g(u)\,, \qquad \int_{\mathcal{S}_{t_1}} (\xi_2)_{(h)}\, g(u_{(h)}) \longrightarrow \int_{\mathcal{S}_{t_1}} \xi_2\, g(u)\,.$$

By similar arguments

$$\int_{Q_{t_1}} (\xi_3)_{(h)} \cdot \nabla g(u_{(h)}) \longrightarrow \int_{Q_{t_1}} \xi_3 \cdot \nabla g(u)\,,$$

for $h \to 0$. This proves the claim. □

A.4 Some properties of the functional spaces used by Maxwell's equations

We recall the definitions (3.56) and (3.58) of the spaces $\mathcal{H}$ and $\mathcal{H}_\mu$.

Lemma A.4.1. If the function μ satisfies the assumption (3.44), the space $\mathcal{H}_\mu(\tilde{\Omega})$ has the following properties:

(1) There exists a positive constant $C > 0$ such that for all $\psi \in \mathcal{H}_\mu(\tilde{\Omega})$,

$$\|\psi\|_{[L^2(\tilde{\Omega})]^3} \leq C \, \| \operatorname{curl} \psi\|_{[L^2(\tilde{\Omega})]^3} \, .$$

The estimate $C \leq c\sqrt{\mu_u/\mu_l}$ is valid, where the constant c depends only on the domain.

(2) The injection $\mathcal{H}_\mu(\tilde{\Omega}) \hookrightarrow [L^2(\tilde{\Omega})]^3$ is compact.

Proof. The compactness result (2) was proved in Picard [1984] for general Lipschitz domains. A proof in a classical analysis setting can be found in Monk [2003]. The estimate (1) is a consequence of the compactness result and can be proved by a classical contradiction argument. Finally, observe that if $\psi \in \mathcal{H}_\mu(\tilde{\Omega})$, there exists by Lemma I.3.6 in Girault and Raviart [1986], a potential A in the space $L^2_{\mathrm{curl}}(\tilde{\Omega}) \cap L^2_{\mathrm{div}}(\tilde{\Omega})$ such that

$$\begin{aligned} &\operatorname{div} A = 0 \, , \quad \operatorname{curl} A = \mu \, \psi \, , \quad \text{ in } \tilde{\Omega} \, , \\ &\gamma_\tau(A) = 0 \quad \text{ on } \partial\tilde{\Omega} \, . \end{aligned}$$

In addition, there exists a positive constant c independent of μ and ψ such that

$$\|A\|_{[L^2(\tilde{\Omega})]^3} \leq c \, \|\mu \, \psi\|_{[L^2(\tilde{\Omega})]^3} \, .$$

We can write

$$\int_{\tilde{\Omega}} \mu \, |\psi|^2 = \int_{\tilde{\Omega}} \operatorname{curl} A \cdot \psi = \int_{\tilde{\Omega}} A \cdot \operatorname{curl} \psi \, .$$

Therefore, by Hölder's inequality

$$\int_{\tilde{\Omega}} \mu \, |\psi|^2 \leq c \, \|\mu \, \psi\|_{[L^2(\tilde{\Omega})]^3} \, \| \operatorname{curl} \psi\|_{[L^2(\tilde{\Omega})]^3} \, .$$

The claim follows. □

Lemma A.4.2. (1) Every vector field $j_0 \in [L^2(\tilde{\Omega})]^3$ such that

$$\operatorname{div} j_0 = 0 \text{ in the generalized sense in } \tilde{\Omega}, \quad j_0 = 0 \text{ a. e. in } \tilde{\Omega}_{nc} \, ,$$

is uniquely representable as $\operatorname{curl} \psi$ with some $\psi \in \mathcal{H}_\mu(\tilde{\Omega})$.

(2) If $\psi \in \mathcal{H}_\mu(\tilde{\Omega})$, then $\gamma_n(\operatorname{curl} \psi) = 0$ in the sense of traces on $\partial\tilde{\Omega}_c$.

Proof. See Druet [2007a] Lemma 2.4, Lemma 2.7 and Lemma 4.2. □

Lemma A.4.3. Let $H \in L^2(0,T;\mathcal{H}_\mu(\tilde{\Omega}))$ be such that $H' \in L^2(0,T;[\mathcal{H}_\mu(\tilde{\Omega})]^*)$, and let $H(0) = 0$. Assume that there exists $\xi \in [L^2(\tilde{Q})]^3$ such that the relation

$$\int_0^T \langle \mu\, H', \psi \rangle + \int_{\tilde{Q}} \xi \cdot \operatorname{curl} \psi = 0\,, \tag{A.25}$$

is valid for all $\psi \in L^2(0,T;\mathcal{H}_\mu(\tilde{\Omega}))$. Then we have $H' \in L^2(0,T;[\mathcal{H}(\tilde{\Omega})]^*)$, and the relation (A.25) holds true for all $\psi \in L^2(0,T;\mathcal{H}(\tilde{\Omega}))$.

Proof. Using Steklov averaging, we can first prove that the relation

$$\int_{\tilde{Q}_{t_1}} \mu\, \frac{\partial H_{(h)}}{\partial t} \cdot \psi + \int_{\tilde{Q}_{t_1}} \xi_{(h)} \cdot \operatorname{curl} \psi = 0\,,$$

is satisfied for all $\psi \in L^2(0,T;\mathcal{H}_\mu(\tilde{\Omega}))$ and all $t_1 < T - h$. We consider an arbitrary $\tilde{\psi} \in L^2(0,T;\mathcal{H}(\tilde{\Omega}))$. For almost all $t < T$, we find a $q_t \in W^{1,2}_M(\tilde{\Omega})$ such that

$$\int_{\tilde{\Omega}} \mu\, \nabla q_t \cdot \nabla \xi = \int_{\tilde{\Omega}} \mu\, \tilde{\psi}(t) \cdot \nabla \xi\,,$$

for all $\xi \in W^{1,2}(\tilde{\Omega})$. The subscript M denotes the subspace of functions with vanishing mean value over $\tilde{\Omega}$. By straightforward considerations, which ensure the Bochner measurability of the mapping $t \mapsto q_t$, we verify that $\psi := \tilde{\psi} - \nabla q$ is a well-defined element of the space $L^2(0,T;\mathcal{H}_\mu(\tilde{\Omega}))$. Therefore,

$$\int_{\tilde{Q}_{t_1}} \mu\, \frac{\partial H_{(h)}}{\partial t} \cdot \tilde{\psi} + \int_{\tilde{Q}_{t_1}} \xi_{(h)} \cdot \operatorname{curl} \tilde{\psi} = \int_{\tilde{Q}_{t_1}} \mu\, \frac{\partial H_{(h)}}{\partial t} \cdot \nabla q\,.$$

Now, we show that $\int_{\tilde{Q}_{t_1}} \mu\, \frac{\partial H_{(h)}}{\partial t} \cdot \nabla q = 0$. For $l \in \mathbb{N}$, we can choose a number $m_l \in \mathbb{N}$, a set $\{\zeta_j\}_{j=1,\dots,m_l} \subset C_c^\infty(0,t_1)$ and a set $\{\phi_j\}_{j=1,\dots,m_l} \subset W^{1,2}(\tilde{\Omega})$ such that the linear combinations $q_l(t,x) := \sum_{i=1}^{m_l} \zeta_j(t)\, \phi_j(x)$ converge to q in the norm of $L^2(0,t_1;W^{1,2}(\tilde{\Omega}))$ for $l \to \infty$. We now introduce the notation

$$(q_l)_{(\underline{h})}(t) := \frac{1}{h} \int_{t-h}^t q_l(s,\cdot)\, d\, s\,.$$

Using integration by parts, and well-known properties of the Steklov averaging, we can write

$$\int_{\tilde{Q}_{t_1}} \mu\, \frac{\partial H_{(h)}}{\partial t} \cdot \nabla q_l = -\int_{\tilde{Q}_{t_1}} \mu\, H \cdot \nabla \left(\frac{\partial}{\partial t} (q_l)_{(\underline{h})} \right) = 0\,,$$

since $H \in L^2(0,T;\mathcal{H}_\mu(\tilde{\Omega}))$. Therefore, we have for all $\tilde{\psi} \in L^2(0,T;\mathcal{H}(\tilde{\Omega}))$ that

$$\int_{\tilde{Q}_{t_1}} \mu\, \frac{\partial H_{(h)}}{\partial t} \cdot \tilde{\psi} + \int_{\tilde{Q}_{t_1}} \xi_{(h)} \cdot \operatorname{curl} \tilde{\psi} = 0\,.$$

The claim follows. □

Appendix B

Equations in dimensionless form

Scaling a problem with typical quantities provides helpful informations about the order of magnitude of different terms appearing in the equations. It is a first important step towards physical understanding and sound mathematical modeling.

For simplicity, we restrict our considerations to the domain Ω_1 occupied by the fluid. We also assume that the material properties of the fluid η, κ, μ, etc. can be considered as constant.

We introduce *typical quantities* for the problem, a distance d (for example the height of the fluid layer), a dominant velocity V (for example the velocity of the crucible rotation), a typical magnetic field strength H_0, a pressure p_0 and a temperature difference β. A caracteristic time is given by $\tau := d/V$. We denote by g the constant of earth acceleration.

Time and space are scaled according to $\bar{T} := T/\tau$ and $\bar{\Omega}_1 := \{\bar{x} \in \mathbb{R}^3 \,:\, \bar{x} = x/d,\, x \in \Omega_1\}$. The unknowns are scaled by $\bar{v}(\bar{t}, \bar{x}) := v(x, t)/V$, etc.

Scaling the system (2.1), (2.7), (3.28) with the help of these new quantities, we obtain a system of equations posed in the set $]0, \bar{T}[\times\bar{\Omega}_1$.

$$\frac{\partial \bar{v}}{\partial \bar{t}} + (\bar{v}\cdot\overline{\nabla})\bar{v} = \frac{\eta}{\rho\, d\, V}\,\overline{\triangle} v - \frac{p_0}{\rho\, V^2}\,\overline{\nabla}\left(\bar{p} - \frac{\rho\, g\, d}{p_0}\,\bar{z}\right) + \frac{\mu\, H_0^2}{\rho\, V^2}\,(\overline{\mathrm{curl}}\bar{H}\times\bar{H}) - \frac{g\,\beta\,\alpha\, d}{V^2}\,(\bar{\theta} - \bar{\theta}_{\mathrm{Ref}})\,\bar{z}\,, \tag{B.1}$$

where $\bar{z}$ denotes the vector $(0, 0, x_3/d)$ and the differential operators $\overline{\nabla}$, $\overline{\mathrm{curl}}$, etc., indicate differentiation with respect to the new coordinates.

Applying the curl operator to the relation (3.28), one can derive for the magnetic field $\bar{H}$ the *equation of induction*

$$\frac{\partial \bar{H}}{\partial \bar{t}} + \frac{1}{\mathfrak{s}\,\mu\, V\, d}\,\overline{\mathrm{curl}}\,\overline{\mathrm{curl}}\,\bar{H} = \overline{\mathrm{curl}}(\bar{v}\times\bar{H})\,. \tag{B.2}$$

Finally, the energy balance has the form

$$\frac{\partial \bar{\theta}}{\partial \bar{t}} + (\bar{v}\cdot\overline{\nabla})\bar{\theta} = \frac{\kappa}{\rho\, c_V\, V\, d}\,\overline{\triangle}\bar{\theta} + \frac{H_0^2}{\mathfrak{s}\, d\, V\, c_V\,\rho\,\beta}\,|\overline{\mathrm{curl}}\bar{H}|^2 + \frac{\eta\, V}{d\, c_V\,\rho\,\beta}\,|\overline{\nabla}\bar{v}|^2\,. \tag{B.3}$$

The different dimensionless numbers appearing in the scaled equations play an important role for the understanding of physical phenomena described by (B.1), (B.2), (B.3). At certain critical values of these numbers, the fluid flow, the magnetic field distribution and the heat transfer phenomena decisively change character.

1. The Mach number $M := V/a$, with the speed of sound $a := \gamma\, p_0/\rho$ ($\gamma = c_p/c_V$ the ratio of the specific heats). It is a measure for the compressibility of a fluid, and is a critical parameter for the validity of Boussinesq's approximation.

2. The Reynolds number $Re := \frac{\rho\, V\, d}{\eta}$. Below its critical value Re_c, the flow is dominated by the viscous forces and *laminar*, beyond Re_c the flow is increasingly inertial, nonlinear and oscillating (*turbulent flow*).

3. The magnetic Reynolds number $Rm := \mathfrak{s}\, \mu\, V\, d$ measures the relative importance of the conductive current to the motion induced current.

4. The Rayleigh number $Ra := g\, \alpha\, \beta\, d^4\, \rho/\kappa\, \eta$ has been proved to be the critical parameter for the onset of thermal instability (see Cap [1972], Volume III, page 134 or Chandrasekhar [1981], page 7-9).

5. The Hartmann number $Ha := \sqrt{\mathfrak{s}\, \mu^2\, H_0^2\, d^2/\eta}$ is an important indicator for how much a flow in influenced by a magnetic field. An increasing Hartmann number (for example increasing magnetic field strength) generally induces an increment of the critical Reynolds number of the flow (see Cap [1972], III, page 127-128), and, for a thermally instable flow, an increment of the critical Rayleigh number (see Cap [1972], Volume III, page 134).

6. The skin depth $\delta := \sqrt{\frac{2}{\omega\, \mathfrak{s}\, \mu}}$, where ω is a typical frequency of the applied alternating current measures how far the magnetic field penetrates the body.

7. The Taylor number $Ta := 4\, \Omega^2\, d^4\, \rho^2/\eta^2$, where Ω is a typical angular velocity, plays an important role in the stability discussion for rotating fluids (see again Chandrasekhar [1981]), comparable to the Hartmann number in the case of electrically conducting fluids. In the context of Problem (P), the Taylor number would appear in equation (B.1) after introduction of the boundary conditions imposed by the rotating crucible and the rotating crystal.

Many dimensionless numbers associated with the names Prandtl, Grashof, Bond, Péclet, and others, are frequently mentioned in the literature in the context of similar problems.

> "It should be pointed out that there is nothing unique in the way these various numbers are defined ; they happen to be the ones which have been chosen ; and in some sense they occur most naturally in certain types of problems"[1].

[1]Chandrasekhar [1981], page 7

Appendix C

Some complements to the model for heat and mass transfer

The model introduced in the paragraph 2.2 is suitable for investigating the influence of applied magnetic fields on the melt flow pattern in a crystal growth furnace. It is also appropriate to giving reliable informations on global heat transfer phenomena in the furnace. The model is however not designed to give precise informations about heat and mass transfer phenomena taking place in the direct neighborhood of the crystallization interface. As a matter of fact, the proposed model

(1) does not treat the crystallization interface crystal-melt as a free material boundary.

(2) neglects the gas flow in the transparent enclosure.

(3) does not treat the boundary melt-gas as a free surface.

(4) neglects to explain how the crystal is actually growing.

The points (1), (3) are obviously of decisive importance if one wants to describe the process of solidification and the meniscus. On the other hand, the approximation (2) of neglecting the convective heat transfer in the gas enclosure is qualitatively well justified, since in general, the convective heat transport is dominated by the radiative heat exchanges in the gas cavity. However, significant loss of quantitative accuracy are observed in certain situations (see Yakovlev et al. [2003], Voigt [2001]).

In the present paragraph, we try to complete the model of paragraph 2.2 to account for the effects (1), (3) and (2).

For simplicity, we may assume that the gas uniformely consists Argon, and that some technique is employed so that evaporation of the melt can be neglected (we think of liquid encapsulation). In the transparent cavity Ω_0 filled with gas, we now have the heat balance in $]0,T[\times\Omega_0$

$$\rho\, c_V\,\left(\frac{\partial\theta}{\partial t} + v\cdot\nabla\theta\right) = \operatorname{div}(\kappa\,\nabla\theta) + 2\,\eta\,D(v,\,v) + (\eta' - 2\,\eta/3)\,(\operatorname{div} v)^2 + \frac{|j|^2}{\mathfrak{s}}\,, \tag{C.1}$$

where ρ, v are the density and the velocity of the gas, and η' its second viscosity (for Argon $\eta' = 0$). Since the gas flow is expected to be turbulent, the Boussinesq approximation cannot be employed to describe the gas motion. The density ρ satisfies the continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\, v) = 0\,. \tag{C.2}$$

Assuming that the gas is not electrically conducting (what for argon is the case), no interaction with the magnetic field need to be taken into account, and the momentum balance takes the form

$$\rho \left(\frac{\partial v}{\partial t} + (v \cdot \nabla) v \right) = \operatorname{div} T + (\rho - \rho_{\mathrm{Ref}})\, \vec{g}\,, \tag{C.3}$$

with the rate of deformation T. We further have $T = S(v) - p\, I$ with the rate of strain S and the pressure p. For compressible Newtonian fluids

$$S(v) := S_{i,j}(v) := \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\, \delta_{i,j}\, \operatorname{div} v \right) + \eta'\, \delta_{i,j}\, \operatorname{div} v \quad (i, j = 1, \ldots, 3)\,,$$

We further need an equation of state. For argon, we can employ

$$p = \frac{R}{n}\, \theta\, \rho\,, \tag{C.4}$$

where n is the molar mass and R is a positive constant.

We denote the total free surface of the liquid by I. We have $I =: I_{gl} \cup I_{ls}$ disjoint, where I_{gl} denotes the interface gas-liquid and I_{ls} the interface liquid-solid, as depicted in Figure C.1.

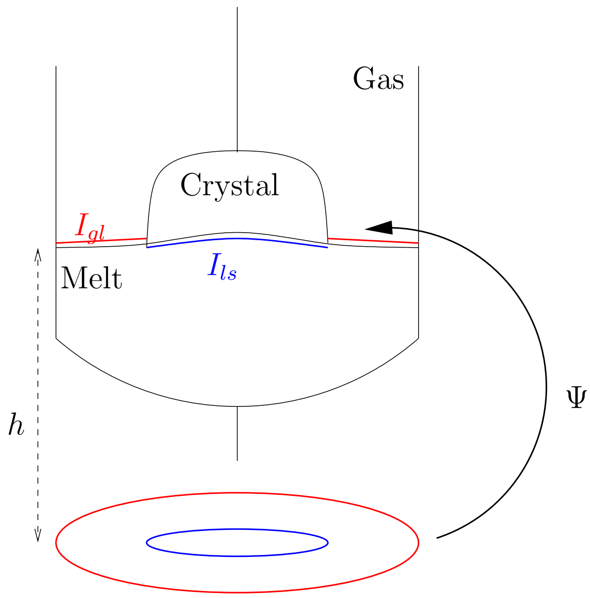


Figure C.1: The parameterization of the free surface I and the height function h.

Boundary conditions On I_{gl}, the heat flux satisfies the radiation condition

$$[-\kappa\, \nabla\theta \cdot \vec{n}] = R - J\,, \tag{C.5}$$

where the radiosity R and the incoming radiation J satisfy (2.10) and (2.13). The Stefan-condition for the heat flux has to be imposed on I_{ls}

$$[-\kappa \nabla\theta \cdot \vec{n}] = L\,V\,, \tag{C.6}$$

where L is the heat released by melting, and V the interfacial velocity in normal direction. The temperature at the interface is described by the Gibbs-Thomson relation

$$\theta = \theta_{\text{eq}}\,(1 + \frac{\gamma\,\mathfrak{k}}{L})\,, \tag{C.7}$$

where θ_{eq} is equilibrium temperature for a flat interface, γ is the coefficient of surface tension, L the latent heat, and $\mathfrak{k}$ the mean curvature.

On material interfaces, we have for the velocity the condition of *conservation of mass* and the tangential continuity

$$[\rho\,v \cdot \vec{n}] = [\rho]\,V\,, \quad [v_\tau] = 0\,, \tag{C.8}$$

where τ stands for the tangential component of v. On the surface I_{gl} where no phase transition is supposed to take place, these conditions simplify to

$$v \cdot \vec{n} = V\,, \quad [v_\tau] = 0\,,$$

with the velocity V of the interface.

In the absence of a rule similar to (C.7) on I_{gl}, the position of the interface has to be determined from the conditions of dynamical equilibrium. The surface force acting in the melt is given by

$$\vec{F}^i_{\text{melt}} = \eta\,D_{i,j}(v)\,\vec{n}_j - p\,\vec{n}_i\,,$$

and the surface force acting in the gas by

$$\vec{F}^i_{\text{gas}} = T_{i,j}\,\vec{n}_j\,.$$

Dynamical equilibrium at the surface I_{gl} will occur if

$$\vec{F}^i_{\text{melt}} - \vec{F}^i_{\text{gas}} = \vec{F}^i_{\text{interface}}\,, \tag{C.9}$$

where

$$\vec{F}^i_{\text{interface}} = -\gamma\,\mathfrak{k}\,\vec{n}_i + \nabla\gamma\,\vec{\tau}_i\,,$$

where the second term accounts for the Marangoni convection in the case that the surface tension is temperature-dependent.

Parameter representation. We now assume that the interface I that separates the melt from the crystal and from the gas is of class C^1 piecewise, and described as the graph of a Lipschitz continuous function ψ. A suitable parameterization domain G is for instance the bottom of the crucible. A function ψ is chosen so that there exists an open subdomain $G_1 \subset\subset G$ such that for the interface I_{ls} separating the liquid and the solid, we have

$$I_{ls} = \{x \in \mathbb{R}^3 \,:\, x_3 = \psi(y),\, y \in G_1\}\,,$$

and the interface I_{gl} between the melt and the gas is

$$I_{gl} = \{x \in \mathbb{R}^3 \,:\, x_3 = \psi(y),\, y \in G \setminus G_1\}\,.$$

We then have that $I_{ls} \cap I_{gl} = \psi(\partial G_1)$. See the figure C.1. According to the Gibb's Thomson law, we have

$$\frac{L\,\theta(\psi)}{\theta_{\mathrm{eq}}} - \tilde{\mathrm{div}}\,\Big(\frac{\gamma\,\tilde{\nabla}\psi}{\sqrt{1+|\tilde{\nabla}\psi|^2}}\Big) = 0\,, \qquad \text{in } G_1\,,$$

where the operators $\tilde{\mathrm{div}}$, $\tilde{\nabla}$ indicate differentiation with respect to the local variables. On the other hand, (C.9) amounts to

$$([T\cdot\vec{n}]\cdot\vec{n})(\psi) - \tilde{\mathrm{div}}\,\Big(\gamma\,\frac{\tilde{\nabla}\,\psi}{\sqrt{1+|\tilde{\nabla}\,\psi|^2}}\Big) = 0\,, \qquad \text{in } G\,.$$

Contact angle conditions at the triple points $I_{ls} \cap I_{gl}$ are reformulated in terms of the jump condition

$$\left[\frac{-\gamma\,\tilde{\nabla}\,\psi\cdot\vec{\tilde{n}}}{\sqrt{1+|\tilde{\nabla}\,\psi|^2}}\right] = \gamma_{GS}\,, \qquad \text{on } \partial G_1\,,$$

with γ_{GS} being the tension of the interface gas-solid, and $\vec{\tilde{n}}$ is the outwards pointing normal to ∂G_1.

On the external boundary ∂G, we assume that the surface I is uniform in height. This can be justified since the contact angle at the boundary of the crucible is less important than in the crystal region. We therefore assume that

$$\psi(t, x_1, x_2) = h(t) \qquad \text{on } \partial G\,, \tag{C.10}$$

with the height function h that describes the height of the melt in the crucible (see the figure C.1).

Conditions for the height function We now want to assume that the crystal has a constant diameter d. We denote by v_p the pulling velocity of the crystal. We denote by h_c :$]0, T[\times G_1 \to \mathbb{R}$ the height function for the interface melt-crystal, and by h_s : $]0, T[\times G \setminus G_1 \to \mathbb{R}$ the height function for the interface melt-gas. Denote by ρ_c, ρ_s the density of the crystal and of the melt, here assumed to be constants.

Let $0 < t_1 < t_2 < T$ be two arbitrary times. We denote by $V_s(t)$ the total volume occupied by the melt at time t. Then,

$$V_s(t_2) - V_s(t_1) = V_1 + V_2 + V_3\,, \tag{C.11}$$

with

$V_1 := \pi\, d^2\, v_p\, (t_2 - t_1)$ is the volume freed by the pulling of the crystal.

V_2 denotes the volume lost by the melt over the solid due to solidification, according to the formula

$$V_2 = \int_{G_1} \{h_c(t_2, x) - h_c(t_1, x)\}\, dx\,.$$

V_3 denotes the volume gained or lost by the melt over the gas due to moving of the interface, according to the formula

$$V_3 = \int_{G \setminus G_1} \{h_s(t_2, x) - h_s(t_1, x)\}\, dx\,.$$

On the other hand, the total volume occupied by the melt can only vary due to a mass flux from the melt to the crystal, and therefore

$$V_s(t_2) - V_s(t_1) = V_2\, \frac{\rho_c}{\rho_s}\,. \tag{C.12}$$

Equating (C.11) and (C.12), we obtain that

$$V_3 = -(V_1 + V_2\, (1 - \frac{\rho_c}{\rho_s}))\,,$$

which means that

$$\begin{aligned}
&\int_{G \setminus G_1} \{h_s(t_2, x) - h_s(t_1, x)\}\, dx \\
&\qquad = -\pi\, d^2\, v_p\, (t_2 - t_1) + (\frac{\rho_c}{\rho_s} - 1) \int_{G_1} \{h_c(t_2, x) - h_c(t_1, x)\}\, dx\,.
\end{aligned}$$

Now, we recall that the function ψ parameterizing the interface can be written as $h(t)+\bar{\psi}$, where $\bar{\psi}$ vanishes on ∂G. We can therefore write

$$\begin{aligned}
&(\text{meas}(G \setminus G_1) + (1 - \rho_c/\rho_s)\ \text{meas}(G_1))\, (h(t_2) - h(t_1)) = \\
&- \int_{G \setminus G_1} \{\bar{\psi}(t_2, x) - \bar{\psi}(t_1, x)\}\, dx - \pi\, d^2\, v_p\, (t_2 - t_1) - (1 - \frac{\rho_c}{\rho_s}) \int_{G_1} \{\bar{\psi}(t_2, x) - \bar{\psi}(t_1, x)\}\, dx\,.
\end{aligned}$$

We divide by $t_2 - t_1$, and we let $t_2 \to t_1$. We obtain

$$\begin{aligned}
&(\text{meas}(G \setminus G_1) + (1 - \rho_c/\rho_s)\ \text{meas}(G_1))\, h'(t) \\
&\qquad = - \int_{G \setminus G_1} V(t, x)\, dx - \pi\, d^2\, v_p - (1 - \frac{\rho_c}{\rho_s}) \int_{G_1} V(t, x)\, dx
\end{aligned}$$

with the interface velocity V. Therefore, the height function h in (C.10) satisfies the ordinary differential equation

$$(\text{meas}(G) - (\rho_c/\rho_s)\ \text{meas}(G_1))\, h'(t) = - \int_G V(t, x)\, dx + \frac{\rho_c}{\rho_s} \int_{G_1} V(t, x)\, dx - \pi\, d^2\, v_p\,. \tag{C.13}$$

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List of Figures

Selbständigkeitserklärung

Ich versichere, dass die Dissertation selbstständig angefertigt wurde, und dass sämtliche Quellen angegeben wurden.

Ich versichere, dass ich die aktuelle Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät der Humboldt-Universität zu Berlin zur Kenntnis genommen habe.